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OXFORD MATHEMATICAL MONOGRAPHS

Editors

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EDDINGTON'S STATISTICAL THEORY

BY

C. W. KILMISTER
King's College, London

AND

B. O. J. TUPPER
University of Exeter

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AUTHORS' NOTE

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PREFACE

The student of Eddington's work will find *Fundamental Theory* an exceedingly obscure, annoying book, in which gems of physical insight are momentarily glimpsed in a thoroughly unworthy setting. It would be impossible in a small monograph to attempt any definitive account of Eddington's ideas, nor do we believe the time to be ripe for this. The present book is only an attempt to point out a possible approach to *Fundamental Theory,* and to fill in just enough details to demonstrate in the last chapter that such an approach can be profitable.

C. W. K.
B. O. J. T.

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TABLE OF SECTIONS

Although our discussions of 'Fundamental Theory' often have to relate different parts of the original book, it is usually possible to refer each of our sections to its most important reference. The following guide is intended to be useful to those with some acquaintance with 'Fundamental Theory', who wish to refer to particular topics.

EST	FT	EST	FT	EST	FT	EST	FT	EST	FT	EST	FT
2.1	1	3.1	7	4.1	14	5.1	5, 49	6.1	23	7.1	37, 38
2.2	2	3.2	8	4.2	15	5.2	(5, 49, 50)	6.2	(23, 24 and algebraic theory)	7.2	38
2.3	1, 2	3.3	9	4.3	16	5.3		6.3		7.3	27, 39
2.4	3	3.4	10	4.4	18	5.4		6.4		7.4	39
2.5	4	3.5	11	4.5	19	5.5		6.5		7.5	39, 40
2.6	5	3.6	12	4.6	22	5.6		6.6		7.6	41–45
		3.7	13			5.7		6.7	23		
						5.8		6.8	24		
						5.9	(50, 51)	6.9	24		
						5.10		6.10	25		
								6.11	26		
								6.12	25		
								6.13	33		

EST, Appendix: **FT**, §§ 6, 17, 21, 34–36, 46–48.

(§§ 20, 27–32 of **FT** are simply discussions of β-factors.)

A reference in brackets denotes sections of original work replacing the sections referred to.

1

SCIENTIFIC THEORIES

1.1. Introduction. THE object of this book is the clarification of Eddington's later researches and, in particular, those published in his posthumous work **Fundamental Theory (1)**, which will be referred to throughout as **FT**. (Numbers in parenthesis refer to the bibliography at the end.) It is fair to say that at the present time these later researches have been rejected by the vast majority of physicists. The authors would be among the first to admit that a great deal of change is needed before Eddington's final works have a satisfactory form. However, they differ from the majority in their opinion that the unsatisfactory nature of his work is probably due to the inherent difficulty of the ideas and that the obscurity is that inevitably associated with the revolutionary. In such circumstances, there seems room for a book which will patiently examine his arguments one by one and try to sift what is valuable from what is false. A recent work of N. B. Slater (**2**), which will be referred to as **S**, has given us a systematic collation of various early manuscripts of **FT**, but Slater was unable in the compass of his work to spend much time on eludication. He has added materially to our knowledge of how Eddington's ideas reached their final form, and in what follows we shall lean heavily on his work, often without acknowledgement.

We shall confine our attention to Chapters I–V of **FT**, that is, to the part described by Eddington as the statistical theory. Our reason for doing this is twofold. Firstly, this part of the book is, to a very large extent, self-contained, and so affords us a convenient starting-point in analysing the whole theory. Secondly, it will be clear from what has been said above that the primary difficulty with Eddington's work is in the underlying ideas rather than in the methods of calculation, and it is in the statistical theory that the ideas are most clearly expressed. This is not to say that even there the ideas are precisely stated. Indeed, they must often be unearthed from ambiguous statements and asides; but the situation is better in the statistical theory. Another way of looking at this distinction is that the most surprising aspect of **FT**, though not in our opinion the most important, is the calculation of certain pure numbers as constants of nature. Chapters I–V purport to show these numbers arising as ratios of numbers of dimensions of certain phase spaces. The other chapters calculate the numbers of dimensions which these spaces will have. Thus we cannot expect, within our limited objective, actually to calculate many pure numbers, but it may be possible to understand how these are calculated by Eddington.

Merely to give a detailed criticism of Eddington's statements would be only of very limited use. Criticism implies a definite, well-understood standpoint, and the reason that Eddington has suffered so much from adverse criticism is that his work is written from a completely different point of view from that of his more orthodox critics. Moreover, he never makes his position clear, and in some of his non-technical works he gives a misleading description of his outlook on physics. This description no doubt agrees with his earlier ways of thinking, but the authors believe that his work can only be understood from a quite different standpoint, and therefore they are forced to the conclusion that consciously or unconsciously in his last years he also had this new way of looking at scientific theories. This chapter will deal with the formulation of this outlook in as much detail as necessary, and then in subsequent chapters we will be able to give a detailed criticism of Eddington's work, paragraph by paragraph. We shall, throughout, try to avoid being merely destructive and where his arguments are at fault we shall attempt to replace them by valid arguments leading to more or less similar conclusions.

In constructing the new point of view in this chapter we shall consider the whole of **FT**, and not merely the statistical theory, so as to be as sure as possible that we have the correct basis for understanding it. Thus this chapter serves also to connect the consideration of the statistical theory in the rest of the book with Eddington's algebraic theory.

1.2. Structural physics. There are a number of experimental observations in physics which can be termed 'structural', and the purpose of Eddington's work is a uniform theory of such observations. One result of such a uniform theory is a particular kind of unification between different branches of physics, since structural observations come from a number of subjects at present separate. There have been several other attempts at such a theory since Eddington's, and these may be mentioned here. Kilmister has considered, in conjunction with E. W. Bastin, certain algebraic structures in physics (**3**, **4**, **5**). The discussions given by Bastin and Kilmister are not clear, and this is partly caused by their having very few examples of these structures available. Newman (**6**, **7**) has formulated a theory which he calls 'entity structure', and it appears that this is an algebraic structure of the type considered in Bastin and Kilmister (**3**). As a result, the connexion of Newman's theory with experiment is of an unusual kind. The nature of this connexion is not clear in his papers, but he says (**7**) that the aim of his theory is not necessarily to discover new or better laws, but to understand the foundations of physics. This statement expresses the aim of such theories in a very suggestive manner, though perhaps not sufficiently precisely.

The existence or not of these algebraic structures, underlying physical theories, can only be properly investigated by means of some technique for working with scientific theories. Such theories in the past have often been regarded simply as hypothetico-deductive systems. Of course, to think of them in this way does help us to understand them to some extent, but if it were the essential property of such a theory to be a hypothetico-deductive system, it would not be clear in what it differed from logic alone. The view which we wish to put forward emphasizes instead the continually developing quality of a scientific theory. New information can enter the system at any time in the form of new experimental facts. The surprising thing which calls for explanation is that it is quite practicable to discuss scientific theories, notwithstanding this continual change. This is true in all parts of science, but it is most clearly exhibited in quantum mechanics.

In quantum mechanics we are confronted with the most remarkable fact of the existence of stable structures—we call them elementary particle systems—which have the property that new experimental data can continue to be described in terms of the same set of stable particles. (We do not for the moment consider the unstable particles—taking the line that the possibility of physical measurement depends on the existence of at least some stable ones.) We speak of such particles as having an 'exhaustible set of attributes' to indicate that the results of indefinitely prolonged experimentation will continue to be referable to the same small number of constants. These constants we then think of as being the numerical measures of mass, charge, etc., of an elementary particle. Probably we are forced so to think of them if we are to preserve any form of quantum mechanics that is continuous with macroscopic mechanics as the systems it studies become larger, but this traditional mode of description certainly obscures the facts we wished to stress by referring to the elementary particles as 'particles with an exhaustible set of attributes'. An approach that, on the contrary, illuminates these facts is as follows:

We consider that as there exist particles with a small number of attributes that by their invariance under different circumstances provide us with our ultimate numerical standards in physics, so there must exist some *invariant parts* of the theory, in some sense which needs to be made precise. For example, Newtonian mechanics, special relativity, and general relativity are theories which all have in common (amongst other things) a $(3+1)$ space-time structure.

1.3. Eddington's structural outlook. Let us now consider the relevance of these arguments to the present book. We wish first to draw attention to the way that the whole of Eddington's work exemplifies our theme by its unorthodox algebraic nature. It is clear from several places in **FT** (and rather clearer from the earlier version (**8**)) that Eddington's use of the E-frame differs both from the ordinary use of unit-vectors, and from the orthodox use of the Dirac matrices. For example (**FT**, pp. 265–6), he speaks of 'an essay in the representation of conceptions by symbolic algebra. . . . We have to express in mathematical symbolism what we think we are doing when we measure things; . . . Having examined critically our conceptual interpretation of the process of measurement, we have to define symbols with properties that correspond precisely to the conceptions introduced.' Again, on p. 139, 'The symbolic coefficient describes the nature of the quantity, "nature" being defined by reference to such concepts as relativistic equivalence, chirality, idempotency, all of which are comprised in the general concept of group structure. We call these *structural concepts*. . . .' (See also the extract from the 1943 draft (**S**, pp. 276–8).)

Eddington tries to avoid the detailed carrying out of such an ambitious programme by using heuristic devices. For instance (**8**, Chapter 1), he introduces a four-dimensional space and considers the transformations of vectors and tensors in this space, although (cf. **8**, § 4.3) it is certainly not physical space-time. The connexion between the original space and physical space-time has been explored to some extent by Kilmister (**9**), but without explaining the significance of the original space. This significance can be better seen from Eddington (**8**), p. 109, where he explains the need for '. . . locking the changes of one characteristic to those of another. . . . We consider the simplest form of tensor interlocking and construct the ideal system to which it would apply. We cannot find any more elementary starting-point than the locking of a covariant wave tensor to a mixed wave tensor; . . .' These tensors are in the original space, and this quotation shows the use of this space as a *model* structure, from which we can construct algebraic concepts (structural concepts). By choosing a four-dimensional space as the original structure, the hope is that the new structure will be related to measurement. Such a heuristic procedure might not be expected to be very useful, but in fact Eddington was able to derive considerable assistance from it.

We have now said enough to justify our description of Eddington's algebraic method as unorthodox and to make fairly explicit the problem of formulating such a structural theory in detail, so that it can be more readily understood. Such a formulation must also take account of the considerable success of the approach (at least as a heuristic method) in Eddington's hands.

The importance of all this for the statistical theory, described in the rest of this book, is simply that the idea of a simpler theory underlying a more complex one (which arises naturally in the algebraic theory) occurs over and over again in the statistical theory. One may say that the statistical theory describes the physical concepts more clearly than the algebraic theory, but it is based on structural concepts derived by Eddington by pondering on the algebraic theory.

The importance of structural considerations has already been pointed out by Bastin and Kilmister (**3**), using the phrase 'fundamental non-metrical physics'. Simple examples of facts in such a non-metrical physics are the number of degrees of freedom of an event (4), of the electromagnetic field (6 if we are dealing with field-strengths) or of the gravitational field (10 in general relativity). In such terms it is already possible to pose problems not yet certainly solved. For example, have the combined electromagnetic and gravitational fields more than 10 degrees of freedom? (This is the point of difference between the Einstein–Maxwell theory (e.g.) and Wheeler's 'Geometrodynamics'.) Wheeler's approach to this question is by means of metrical physics, but Bastin and Kilmister showed (**4**) that there are some structural problems which can be answered by a structural theory alone.

An implicit realization of the importance of algebraic structures in physics occurred already in 1938 in Dirac's discussion (**10**) about the dimensionless physical constants. (We are not here concerned with the truth of Dirac's hypothesis in that paper.) Dirac observed that the fundamental dimensionless constants fall into three classes:

(i) those like the proton-electron mass-ratio and the fine-structure constant which are of order 1;

(ii) those like the force constant (ratio of electrical to gravitational force between proton and electron) and Hubble's constant in atomic units which are of order 10^{39};

(iii) those like the ratio of the total mass of the universe to that of the hydrogen atom which are of order 10^{79}.

In order to arrive at such a generalization, Dirac permitted himself to prepare the experimental material by replacing any dimensional constant which was less than 1 by its reciprocal. Otherwise, he considered just those constants which presented themselves naturally. Now, in order that any result of Dirac's form could be anything but a most remarkable coincidence we must have some very strict limitation on the constants which can enter. For example, it might appear that by rewriting our equations in a different form we could make the constant e^{m/m_0} an important one, instead of the original mass ratio, but this constant naturally would not fall into any of Dirac's groups. Thus, the system for which Dirac's results hold must be a sufficiently simple one to preclude such mathematical operations.

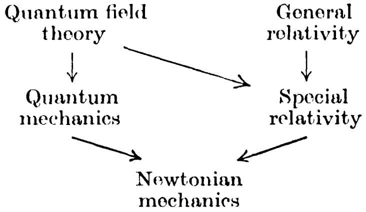
Again, Bohr and his school in Copenhagen, as a result of a detailed analysis of the conditions of measurement in quantum-mechanical systems, come to the conclusion that the uncertainty principle implies that no single model can be adequate to describe the state of an elementary particle. Such a conclusion requires a considerable modification of our way of thinking about scientific theories, and yet is consistent with the possibility of formulating our knowledge of the particles in such a way that the numerical values of the atomic constants are assimilated directly to some (algebraic) structures and only derivatively to properties like mass, spin, and charge.

1.4. The structural assumptions.

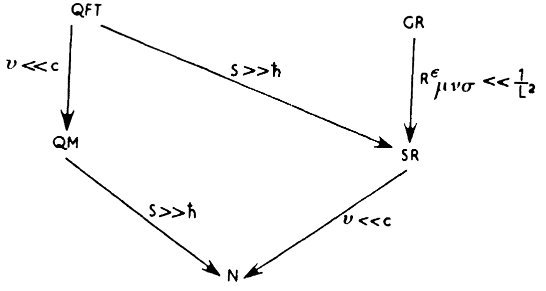
We now proceed to outline a theory of the invariant algebraic structures which makes it possible for the numerical constants arising from the structures to be identified with atomic constants.

In the first place we suppose that to a given invariant part of physical theory there corresponds a set of relevant experiments (those experiments which, we would normally say, have meaning in that part of the theory, are describable in it, and the results of which are predictable to a certain degree of accuracy in it). To take an example, consider the theory defined to be that part of Newtonian mechanics which follows from Newton's first law and the laws of elastic impact. Such a theory contains most of the experiments in a game of billiards. If we consider, however, the experiment of simultaneously shooting three perfectly elastic billiard balls up the medians of an equilateral triangle to converge at the centre, we find that an indefinitely small error in the time of shooting produces a large change in the result. This experiment, then, belongs to a more complex theory (belongs in fact to some theory which treats elastic properties explicitly).

We can then imagine sequences of theories, Σ_1, $\Sigma_2, \ldots$, each sequence arranged in order of complexity (we may write $\Sigma_r > \Sigma_s$ if every experiment of Σ_s belongs to Σ_r but not vice versa). In such a sequence the more complex theories predict the results of more experiments and so are more open to amendment as the result of an inflow of new information. The simplest members of the sequence are least open to such development and in this are nearer to mathematics than to usual physical theories. It is not easy to give a precise example of this state of affairs because actual scientific theories are not well defined, but the following table (where arrows denote the ordering by inclusion) will give some idea.



The process of decreasing the number of variables in an experiment which belongs to one theory so that the amended experiment belongs to a simpler theory is called *abstraction*. In fundamental physics this is often (not always) achieved by considering an experimental configuration in which certain fundamental constants become negligible. For example, in our previous table, we may mark in the conditions for the abstractions shown:



(where S is a typical amount of action and L a typical length).

We remarked above that the simpler systems are less open to amendment as a result of new information, so that the simplest ones of all will simply be formal systems of a certain type. Therefore, our assumption has to be amplified to say that the process of abstraction can be extended until the results of the abstraction are certain formal systems.

The novel part of the present approach is the hypothesis that the experimental contact of each structure has the same degree of physical reality (the orthodox view would be that the most complex structure gives greatest experimental contact, the simpler structures being merely simpler theoretical models). Naturally, it is easier to design experiments in a very complex abstract structure, and in the simplest abstract structures that we know it is difficult to devise any experiments at all. But any experiment is only completely specified if we know what structure it is framed in. The experiments of elementary dynamics contain smooth planes, elastic particles and so on; and it is misleading to say simply that these are not available in the world. It is better to say that elementary dynamics has just as real an experimental contact as any other theory, only we must prepare experiments in such a way that friction and so on do not affect the results (e.g. Atwood's machine).

It would be generally agreed that some parts of physics are more 'fundamental' than others, although this term is admittedly a vague one. In so far as the term can be given a precise meaning, we can make our hypothesis in a form which is *refutable* (it therefore becomes an ordinary scientific hypothesis):

> *The simplest abstract structures describe the most fundamental experiments*

The more complex structures have then the property that our knowledge of them is less complete, but must be gradually added to by further experiment in the structure. At any one time one could think of a complete structure, but it is misleading to do so. We should really think of these structures as continually changing in time. In Bastin and Kilmister (**4**) this point gave rise to the use of Brouwer's mathematics, but it was not made clear which parts of the theory really required the intuitionist uncertainty and in which parts we could retain the more convenient classical methods. We are now able to make an explicit assumption about the abstract structures, which was implicit in Bastin and Kilmister (**3**), and to relate the necessity for this assumption to the Brouwerian characteristic of the complex abstract structures. The justification for this assumption is the agreement with experiment of the results to which it leads (it is, therefore, an assumption of the usual kind in science):

Assumption 1. Given the partially ordered set of structures Σ_r, there exists a mapping H onto a corresponding set of *algebraic* structures S_r,

$$H\colon \Sigma_r \to S_r,$$

such that

(i) Each Σ_r determines a unique S_r.
(ii) If $\Sigma_r > \Sigma_t$ then $S_r \geqslant S_t$.
(iii) The S_r can be regarded as invariant while the Σ_r change with new information.

This property of the Σ_r is called (for reasons which will become clear) that of being essentially algebraic and the structures S_r were called theory-languages by Bastin and Kilmister (**3**).

The use of a different letter for the structures S_r is not meant to imply any less degree of physical reality, but these structures are sufficiently simple for us to have a good understanding of them, and the different letter is used to imply that their Brouwerian characteristics are negligible. This is why we can treat them as mathematical structures ('formal systems'). We can then make assumption 1 more precise as follows:

Assumption 2. If $\Sigma_r \to \Sigma_t$ is an abstraction (in the sense of this section), then $S_r \to S_t$ is a homomorphism.

The particular structure theory which we devise then depends on assumptions about the nature of the formal systems used as theory-languages.

It will be as well to make clear at this point the way in which the idea of a theory-language assists us in understanding Dirac's hypothesis, although we are not here concerned with finding fundamental constants. It would be commonly agreed that some measurements are physically significant in themselves in a way that others are not. It is further often maintained that dimensionless constants are those of greatest physical significance. If the abstract structures provide the framework in which alone consistent measurement is possible, and if we suppose that the physically most significant measurements are logically the most primitive, it is natural to expect the dimensionless constants to be identified with some numerical invariants of the simplest abstract structures.

1.5. Remarks on the assumptions. The assumptions made so far are of a different kind from those usual in a scientific theory. The importance of the assumptions of the existence and physical reality of the abstract structures is that no experiments can exist independently of these structures. The usefulness of the assumption 1 then arises from the fact that it is impossible to be completely precise about the more complex structures Σ_r, because of their Brouwerian qualities. We can, however, deal with some of the properties of the abstract structures by means of the algebraic structures S_r. We have to proceed in a tentative manner; we formulate such an S_r directly, probably in a way which involves heuristic arguments, and we then discuss what features of the complex unformulated Σ_r can appear in S_r, and how they appear. The agreement with experiment will then show whether or not our arguments leading to S_r were correct. For example, Bastin and Kilmister (**3**) used certain symmetry conditions which they believed to define the simplest of the canonical structures described above. They then proved that such simple theory-languages were Abelian groups in which every element was of order 2. The simplest such group, which possesses the symmetry property non-trivially, is Klein's quadratic group.

It is, then, clear that such an S_r is in many ways like an ordinary scientific theory, but it differs in certain respects:

(*a*) Once it has been formulated there is no longer any room for additional assumptions, apart from axioms of identification. Thus the theory must agree with experiment as it is, and there is no possibility of slight modifications to 'improve agreement'. Such 'improved agreement' can only arise because the experiment in question is in a more complex abstract structure, and the improvement results by formulating this more complex structure.

(*b*) The identifications which we must make to compare the theory with experiment must often be performed using orthodox terms in a somewhat generalized sense. This is because the structures S_r are very fundamental, so that one concept in S_r may correspond in the mapping $\Sigma_r \to S_r$ to several in Σ_r, and so to several related concepts in the orthodox theory (which gives the names to concepts). This gives rise to a danger of confusion in the exposition which is not present in expounding an ordinary scientific theory.

The peculiarly rigid nature of the S_r theories, noted above, at once brings to mind the similar state of affairs in general relativity. This suggests that general relativity is an abstract structure Σ_r which is in *some* sense nearer to its S_r than other theories; in this sense we could say that general relativity was a very simple theory. This description of general relativity as a simple theory is in agreement with the usual view which emphasizes its four-dimensional nature and its dependence on invariance requirements of an obviously epistemological nature. It makes it clear that general relativity is a theory which has a great deal to tell us about abstract structures. Eddington seems to have been among the first to notice this fact (for instance in (**11**), first part of § 96, concluding part of § 99, and especially § 103), but he was unable to make much progress from this standpoint. (Some early drafts of **FT**, notably that given in **S**, pp. 68–72, show that he retained some hopes of doing so for a long time.) The appearance of Dirac's wave equation for the electron showed Eddington that the S_r for general relativity was not the only one, and that the Dirac matrices described another one. The difficulty is now that the corresponding Σ_r is much farther away and it is in Σ_r that the conventional interpretations are made.

One way which Eddington adopts in **FT** to try to overcome this difficulty is by using β-factors ($\beta = \frac{137}{136}$). From what we have said above it is obvious that we cannot accept the discussion of β-factors given by Eddington. Probably he intends them as a device to avoid the problem of formulating the more complex abstract structure which some experiments require. But we have no means of telling how good an approximation such a device is.

It is instructive to compare the present approach with an earlier attempt of Küssner (**12**) to formulate certain abstract structures in physics. This attempt of Küssner's is vitiated by his using, as an experimental check on his assumptions, a system of physics which is not only unorthodox but almost certainly wrong. Küssner's work, therefore, forms an interesting example for us of how it is possible to formulate abstract structures wrongly (a state of affairs which would not be expected from reading Eddington's work). Küssner's argument is as follows. He tries to relate the difficulties in physics with the crisis in the foundations of mathematics by looking on this crisis as one which arises from extending the number system. The extensions of the number system in physics which he considers are quaternions and their direct products, the even Clifford algebras. One of the results which he claims to deduce (**12**, p. 33) is the four-dimensional nature of space and time. The argument is that the higher algebras have homomorphisms on to quaternion algebra which is a division algebra. He then proceeds (pp. 150–3) by an analogy with the usual dimensional analysis to set up a means of calculating dimensionless constants from certain finite groups associated with the Clifford algebras. He believes that the algebra which is the direct product of n quaternion algebras enables one to calculate n fundamental constants. It is possible to see a certain vague connexion between Küssner's methods and Eddington's, but because of his most unfortunate physical identifications (pp. 53–149) his theory leads nowhere.

1.6. Conclusion. One possible way of proceeding from this point would be to formulate precisely the simplest algebraic structures involved in physics. We have not any straightforward means of doing this; we must proceed by trial and error, taking account of the symmetries which the structures must have and trying to relate the structures to the way we make measurements. Such attempts have been made by Bastin and Kilmister, and by Newman. There is no doubt that this would be the most fundamental way of tackling the problem, but it would be a very lengthy one, and within the compass of this book we could not hope to reach the degree of complexity required to criticize Eddington's work. In any case this path leads directly to the algebraic theory and only in a roundabout way to the statistical theory. Accordingly, we shall instead try to keep in our minds the general idea of a simple abstract structure. We may think of it as a systematic model-making of theories, and from this point of view we shall carry out the page-by-page criticism of Eddington. It will not be necessary in the course of doing this to refer to this basic idea at each stage; it will often be taken for granted. We believe that Eddington also took this idea for granted in his later work. In the following chapters we shall deal, in more or less the same order as Eddington, with Chapters I–V of **FT**, but omitting those parts exclusively concerned with β-factors. Thus the theory described will be cruder than Eddington's, but also simpler to understand.

We conclude this chapter by describing the plan we have worked from in preparing this book. The remainder of this chapter is only of interest to readers who are already familiar with **FT**. Let us begin by making a short summary of **FT**, dividing the statistical theory up into its various logical parts. The following chart shows the very complex interconnexions between the various sections of **FT**. The arrows on the chart imply a principal logical dependence, though naturally there would also be lesser uses of one section in the argument of another which we can probably ignore at first. One thing which emerges from this chart is the anomalous division into chapters which is employed by Eddington. Thus it would be much more natural for section 23 to begin Chapter 3 instead of section 24, and we find from Slater's book that this was in fact the case in an earlier draft.

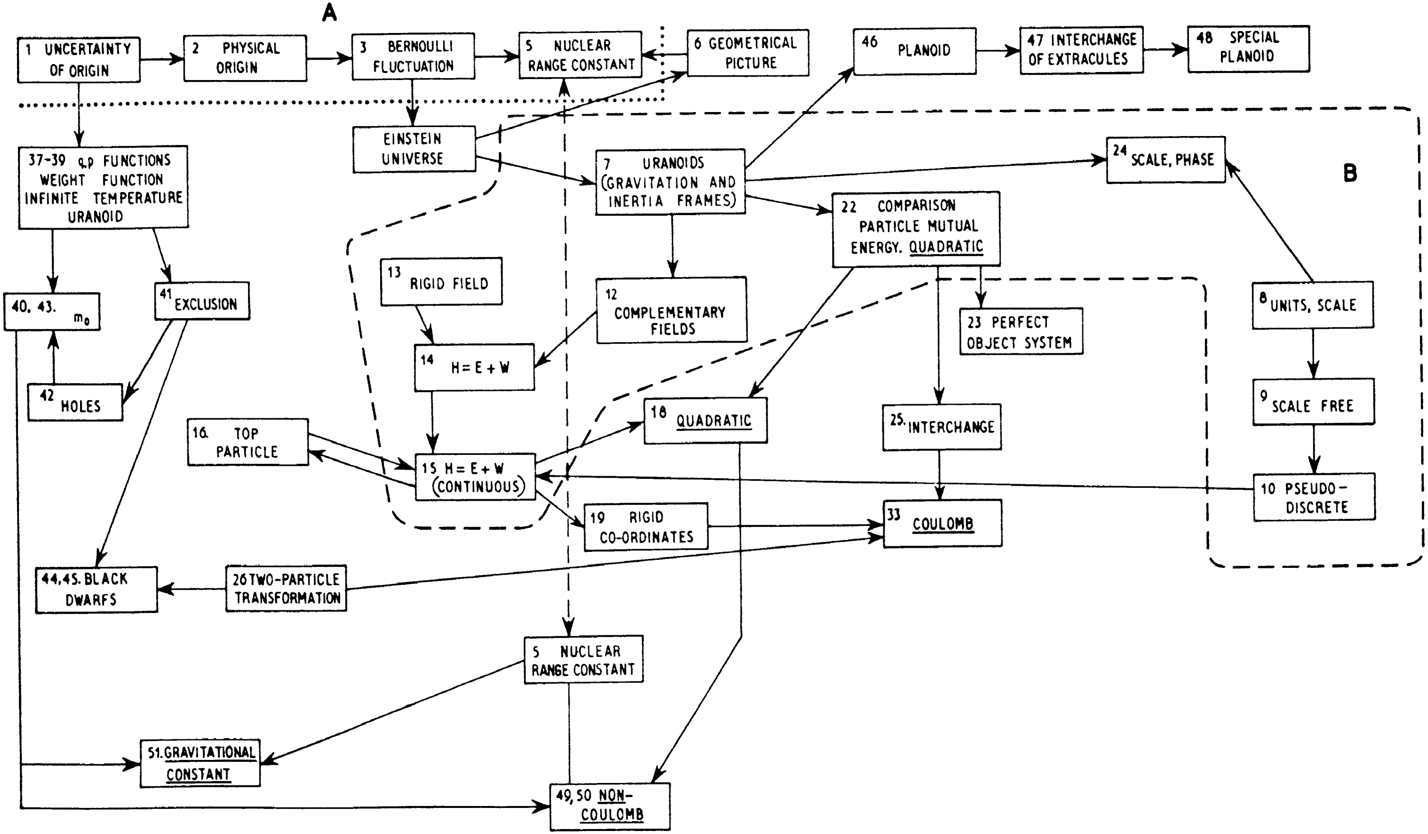

A
B
1 UNCERTAINTY OF ORIGIN
2 PHYSICAL ORIGIN
3 BERNOULLI FLUCTUATION
5 NUCLEAR RANGE CONSTANT
6 GEOMETRICAL PICTURE
46 PLANOID
47 INTERCHANGE OF EXTRACULES
48 SPECIAL PLANOID
EINSTEIN UNIVERSE
37-39 q,p FUNCTIONS WEIGHT FUNCTION INFINITE TEMPERATURE URANOID
7 URANOIDS (GRAVITATION AND INERTIA FRAMES)
24 SCALE, PHASE
22 COMPARISON PARTICLE MUTUAL ENERGY. QUADRATIC
13 RIGID FIELD
12 COMPLEMENTARY FIELDS
40, 43. m_0
41 EXCLUSION
8 UNITS, SCALE
23 PERFECT OBJECT SYSTEM
14 H = E + W
42 HOLES
9 SCALE FREE
18 QUADRATIC
25. INTERCHANGE
16. TOP PARTICLE
15 H = E + W (CONTINUOUS)
10 PSEUDO-DISCRETE
19 RIGID CO-ORDINATES
33 COULOMB
44,45. BLACK DWARFS
26 TWO-PARTICLE TRANSFORMATION
5 NUCLEAR RANGE CONSTANT
51. GRAVITATIONAL CONSTANT
49,50 NON-COULOMB

FIG. 1.

Again, section 33 follows directly on section 26, and this again corresponds to an earlier draft in August 1943. One wonders whether Eddington was led by his unfortunate circumstances to try to fit in sections somehow or other even when the logical order was interrupted. We shall already have made substantial progress towards understanding **FT** when we have sorted out the logical dependence of sections in this way. Another thing that we can observe from the chart is that some chains of development may still be preserved, even after we have shown others to be fallacious.

The rather complex chart can be simplified in the following way. Firstly, we notice a unifying principle in all the arguments employed in the statistical theory; that is the principle that we must take account of the environment. In the terms which we have already defined, we can say that the whole of Eddington's investigations consist of determining the relationship between a simpler and a more complex abstract structure corresponding respectively to the system concerned and the system together with its environment. Now, these investigations fall into the following parts. There are two basic arguments which we shall call 'A' and 'B'. 'A' consists of sections 1 to 5 of **FT** and is fully dealt with in our Chapter 2. 'B' is the longest complete argument to be found in the statistical theory and comprises roughly sections 7 to 15 of **FT** together with section 22. This is described at length in our Chapters 3 and 4. There are then three secondary investigations which are based on these fundamental ones. Of these, the discussion of the non-Coulombian potential is based mainly on 'A' and is found in our Chapter 5. This depends, however, in one respect on the investigations of the proton-electron mass ratio and the Coulomb potential which are based only on 'B' and are found in our Chapter 6. Lastly, the discussion of exclusion and of the nature of proper mass (Mach's principle) with the determination of the constant of gravitation. These are based mainly on 'A' and are found in our Chapter 7. The remaining sections of **FT** are disconnected fragments which we discuss in an appendix. The following charts summarize this description.

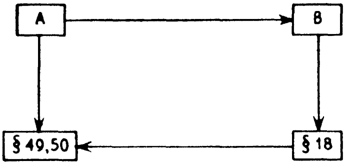


FIG. 2. Non-Coulombian potential.

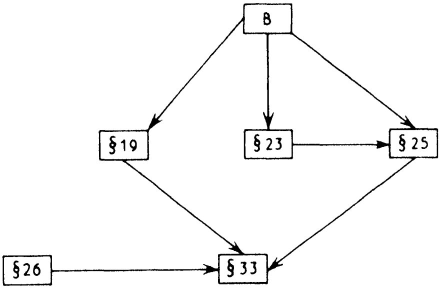


FIG. 3. Coulombian potential

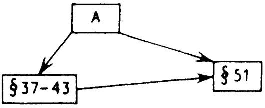


FIG. 4. Proper mass.

2

EDDINGTON'S STATISTICAL THEORY

2.1. WE will now describe argument 'A' of Eddington's theory which consists of sections 1 to 5 of **FT**.

The 'qualitative physics' assumed by Eddington is the relativity principle and Heisenberg's uncertainty principle, both of which are used in very primitive forms. The relativity principle is used to justify the statement that the observed coordinates are relative coordinates of two physical entities. The uncertainty principle requires that a particle can only be assigned a probability distribution of position and velocity. The combined principle used by Eddington is that observed coordinates are relative coordinates of two physical entities, both of which have uncertainty of position and momentum in the geometrical frame; the same considerations applying to other observables.

It is not possible to develop the theory of the Lorentz transformation from Eddington's use of the relativity principle since the probability theory involved is not Lorentz invariant. Eddington was strongly opposed to the view that the equations of quantum theory should be Lorentz invariant and he put forward arguments (**13**, **14**) to support his view.

It follows from the combined principle that an observable coordinate is not measured from the geometrical origin of the auxiliary mathematical frame, but from a physical origin which has an uncertainty of position in the geometrical frame. The geometrical origin may be used but is eliminated in the final calculation of observationally verifiable results.

Suppose we have a system of particles with coordinates (x_r, y_r, z_r) $(r = 1, 2, \ldots)$ in the geometrical frame and suppose the geometrical coordinates of the physical orgin are (x_0, y_0, z_0). Both sets of coordinates are unobservable, but the relative coordinates

$$(\xi_r, \eta_r, \zeta_r) = (x_r - x_0, y_r - y_0, z_r - z_0)$$

are observable. The equations of wave mechanics use a 'physical origin', since they treat coordinates and conjugate momenta as observables. However, the frequency function $f(x_0, y_0, z_0)$ of the physical origin is not specified and knowledge of $f(x_0, y_0, z_0)$ is necessary in order to make the transformation of coordinates from (x_r, y_r, z_r) to (ξ_r, η_r, ζ_r). Some of the equations of quantum theory can only be valid for some particular origin, since they are not of a form which would be invariant for arbitrary changes of f, so we require to know what is the origin employed and what is the frequency function.

2.2. Eddington argues that the physical origin must be defined as the centroid of a large number, N, of particles with a spherically symmetric probability distribution since the form of the probability distribution of this centroid (a Gaussian distribution) does not depend on the probability distribution of the individual particles, subject to certain conditions which are ordinarily fulfilled. Amongst these conditions is that the individual particles are non-interacting, so that we may suppose that either they are particles in a simple theory which describes neither forces of gravitation nor electricity, or that the particles are hydrogen atoms and the gravitational fields are neglected. The neglect of the gravitational fields will be valid if we consider the particles in an Einstein universe, as Eddington does, or in some other curved space. The fields are then represented by the curvature, which does not affect the distribution except through the geometry. This seems to be the intended interpretation, though the possibility of describing unstable particles, such as mesons, is ruled out.

Throughout **FT** the physical origin employed is related to the geometrical origin by a Gaussian frequency function of the form:

$$f(x_0, y_0, z_0) = (2\pi\sigma^2)^{-\frac{3}{2}}e^{-(x_0^2+y_0^2+z_0^2)/2\sigma^2}, \tag{1}$$

where the standard deviation, σ, is known as the 'uncertainty constant'. This uncertainty constant plays an important part in Eddington's theory, since it puts the scale into the physical frame, and one of the main problems in **FT** is to investigate the way in which σ is related to various structures in the physical frame.

The laws and constants which Eddington derives are valid only for measurements referred to the physical frame defined by (1). There is no guarantee that this frame will be the one in which the current quantum equations are valid. Eddington states that f must have a Gaussian form for these frames to be the same, but his argument is difficult to follow.

2.3. We will now present a slightly amended form of some of the arguments used in the two previous sections.

The basis of relativity theory is the recognition of the necessity for a coordinate system in describing physical phenomena. From this basis the theory then goes on to find how to ensure that we are not merely investigating properties of the coordinate system employed. It does this by considering a set Σ of coordinate systems, and defining the group, $G(\Sigma)$, of all transformations $S \rightarrow S'$, where S, S' belong to Σ. A first necessary requirement that we are investigating physically significant properties is then that they should be true in all the coordinate systems of Σ. Since these properties are to be expressed by numbers and relations between numbers, these numbers must transform under a *representation* of the group $G(\Sigma)$. This condition is a general formulation of the 'principle of relativity'; the usual tensor theory then follows. However, here we are concerned with only the initial step in this derivation, that is, with the recognition of the necessity for a coordinate system.

The principle of relativity is a necessary condition of physical significance of properties, but it is not a sufficient one. This fact has been shown by quantum theory. A possible basis for quantum theory, the uncertainty principle, can be expressed in the following form, which has the advantage for us that it is intimately connected with the last section:

The numbers describing a physical system, in the sense of the last section, can be divided into classes of conjugate pairs. It is false to suppose that both members of a class can be known exactly. If one member is known to a certain uncertainty, the minimum uncertainty of the other is specified by the uncertainty relation.

These uncertainties must, of course, be defined accurately in terms of probability distributions. When this has been done, the usual formulation of quantum theory can be built up. As with the relativity principle, we are concerned only with the first step in this derivation. We shall consider, in particular, the position of particles. The conjugate variate is then related to the momentum; however, the form of the restriction which is sufficient for our purposes is simply that the position can only be known with some uncertainty, that is, that a probability distribution is given for the position.

If we are to consider the positions of a number of particles relative to a given coordinate system, each position must then be specified by a probability distribution. It is convenient to avoid the consideration of so many independent distributions by recalling that the uncertainty was introduced merely to conform to the uncertainty relation. We therefore introduce an uncertain origin specifying a coordinate system K; we are then considering a different kind of coordinate system from those understood in the earlier paragraphs, and the transformation $S \to \mathrm{K}$ does not belong to $G(\Sigma)$. We may suppose the coordinates of a particle relative to K to be known exactly, since the uncertainty has been provided, once and for all, by the origin. It should be noted here, however, that although (x^i), (y^i) may be exactly known coordinates of particles X, Y relative to the coordinate system K (here supposed rectangular cartesian) the distance XY will not be known to be exactly $\left[\sum_i (x^i - y^i)^2\right]^{\frac{1}{2}}$, for if we consider both X and Y allowance must be made for them to have independent probability distributions.

It is necessary to be able to specify the transformation $S \to \mathrm{K}$, and, in order to do this, we must know the probability distribution of the origin. We are here faced with a difficulty, since we require to know the frequency function $f(x_0^i)$, say, of the position (x_0^i) of the origin of K relative to S. Now f cannot be found by observation, since the (x_0^i) are not observables. On the other hand, the coordinates postulated in the dynamical equations of quantum theory are assumed to be observable, and must therefore be measured relative to K. The equations are certainly not invariant under arbitrary changes of the function f, so that they must imply a special choice of probability distribution. The special choice which must, in fact, have been made by quantum theory is suggested by the Central Limit Theorem (**15**, p. 180). A consequence of this is that the centroid of a large assembly of non-interacting particles has a Gaussian distribution, no matter what distribution (within certain limits) the particles may have. If there are N particles, each with standard deviation κ, then the standard deviation of the centroid is $\kappa/\sqrt{N}$. Thus, if we postulate the centroid of N particles as (x_0^i), the function is determined. There seems no other equally simple choice of definition. This fact and also the numerical agreement of the results found by Kilmister (**16**) with quantum-mechanical experiment suggests that this is the choice made by quantum theory.

There remains only the determination of the standard deviation, κ, of each particle; but we will leave this for the next section, when we will consider Eddington's derivation of σ as given in **FT**, § 3.

Before proceeding to this, let us consider the requirement imposed by the Central Limit Theorem that the particles are non-interacting. It will clearly be illegitimate to allow charged particles to enter, and so we take the particles as hydrogen atoms. This means that we are restricting our attention to a particularly simple abstract structure in which all the particles are taken as equal. However, there is still gravitational force between the particles, but this may be disregarded if we assume the particles to be in a space-time manifold of general relativity; the gravitational field will then be represented in the geometry of the space. (Eddington seems to have been unaware of the necessity for non-interacting particles in **FT**.) The use of a curved space raises a difficulty in the definition of the centroid in a non-invariant manner. However, we require the idea of a centroid to define the origin of a certain coordinate system, and we accordingly use a coordinate system of the *same* kind to calculate the centroid.

2.4. Returning to **FT**, we find that in § 3 Eddington wishes to consider a system of particles which is in self-equilibrium so that the probability distribution is steady. This is a uniform distribution filling a hyperspherical space, i.e. an Einstein universe. In order to use the finiteness of the Einstein universe, Eddington considers the problem of a set of N particles of which n are measured to be in a volume V. If n_0 is the expectation value of n and $n = n_0+y$, then y has the frequency function

$$f_N(y) = \{2\pi n_0(1-n_0/N)\}^{-\frac{1}{2}}\exp\left[\frac{-y^2}{2n_0(1-n_0/N)}\right]. \qquad (2)$$

The limiting form when $N/n_0 \to \infty$ is called the ordinary fluctuation and corresponds to the infinite Euclidean space of classical theory. The frequency function in this limiting case has the form

$$f_\infty(y) = (2\pi n_0)^{-\frac{1}{2}}\exp[-y^2/2n_0]. \qquad (3)$$

The standard deviations of (2) and (3) are $\sigma_t = (n_0-n_0^2/N)^{\frac{1}{2}}$ and $\sigma_0 = n_0^{\frac{1}{2}}$ respectively, and we can introduce another Gaussian distribution, which Eddington calls the extraordinary fluctuation, with standard deviation

$$\sigma_e = (n_0^2/N)^{\frac{1}{2}}$$

and frequency function

$$f_e(y) = (2\pi n_0^2/N)^{-\frac{1}{2}}\exp[-Ny^2/2n_0^2], \qquad (4)$$

so that the ordinary fluctuation can be regarded as the resultant of the extraordinary and the actual ones. The standard deviations are connected by

$$\sigma_0^2 = \sigma_t^2+\sigma_e^2.$$

The ordinary fluctuation arises from the finiteness of N, which also gives us an Einstein universe instead of a flat space, so the passing from classical to relativity theory leads to the curvature of space and the introduction of the extraordinary fluctuation. Eddington sets out to show that the two are really the same thing.

Eddington has already shown that we are dealing with an Einstein universe, since we are considering a system of N particles which is in self-equilibrium. Now an Einstein universe differs from a flat space only in the radial direction, i.e. a flat space has a metric

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 - c^2\,dt^2,$$

while an Einstein universe has the metric

$$ds^2 = \frac{dr^2}{1 - r^2/\mathrm{R}_0^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 - c^2\,dt^2,$$

where R_0 is the radius of the Einstein universe. Hence in these coordinates the finiteness of an Einstein universe, whether described in terms of R_0 or N, is shown only by its effect in the radial direction.

If we consider a point whose coordinate is r, the difference between its physical and geometrical coordinates consists of:

(a) a fluctuation with standard deviation σ in all directions, due to the uncertainty of position of the physical origin, and

(b) a fluctuation with standard deviation $\sigma_\epsilon r$ in the radial direction only, due to the uncertainty of the scale of measurement of r. σ_ϵ is the standard deviation of some linear scale factor ϵ. Hence, we have that the resultant standard deviation is σ in the transverse direction and $(\sigma^2 \pm \sigma_\epsilon^2 r^2)^{\frac{1}{2}}$ in the radial direction. For this latter quantity, Eddington subtracts the two standard deviations, since he considers σ to be an ordinary fluctuation while $\sigma_\epsilon r$ is an extraordinary fluctuation. We will show that this interpretation is correct, but we will beg no questions and retain the dual sign for the moment.

Eddington calls these standard deviations in the radial and transverse directions the 'local uncertainty' of the physical reference frame, and assumes that the local uncertainty in a given direction serves as a suitable standard of length measurement. (This is discussed more fully in **FT**, § 4.) Assuming that this is possible, we have that the radial and transverse elements of a line-element reckoned in the σ-metric, i.e. the metric in which lengths are measured by local uncertainty, are proportional to $dr^2/(\sigma^2 \pm \sigma_\epsilon^2 r^2)$, $r\,d\theta/\sigma$, $r\sin\theta\,d\phi/\sigma$ so that the space-like part of the metric has the form

$$ds^2 = \frac{dr^2}{1 \pm (\sigma_\epsilon^2 r^2/\sigma^2)} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{5}$$

Now we know that when we deal with a system of N particles we have an Einstein universe, and if we assume that the local uncertainty serves as a standard of length measurement, then when we apply this to N particles, we must have that (5) is the space-like part of an Einstein universe. Hence for the σ-metric to represent an Einstein universe we must have that the time-element is proportional to $ic\,dt/\sigma$, and that we must take the negative sign in (5), so that

$$\sigma = \mathrm{R}_0 \sigma_\epsilon, \tag{6}$$

where R_0 is the radius of the Einstein universe. The condition on the time-element is ignored by Eddington, but if we consider only the space-like coordinates as he does, then we will not have an Einstein universe. The time coordinate is as important as the space coordinates and should not be left out of consideration.

If $\mathrm{R}_0 \to \infty$, which is the same as $N \to \infty$, then $\sigma_\epsilon \to 0$ and the Einstein universe becomes a flat space. Hence, since $\sigma_\epsilon r$ is a fluctuation which arises only through the finiteness of N, it is, in fact, the extraordinary fluctuation, so that σ is the ordinary fluctuation since it is combined negatively with $\sigma_\epsilon r$.

To find the value of σ_ϵ, Eddington considers the extraordinary fluctuation of the particle density (4) as existing alone. The validity of this is doubtful, and we propose to find the value of σ, without the intermediate step of finding σ_ϵ, by following the arguments we put forward in section 2.3.

It is first necessary to remark on the exact status of the standard deviation κ as a physical constant. It is the standard deviation of the distribution of a single particle. This is not, of course, meant to imply that there is one uniquely determined distribution. Rather we must define, by some precise process, a distribution which it is natural (in some sense) to consider for a particular space-time. This is clearly the most difficult part of the theory. We will here put forward one definition of a distribution and the resulting standard deviation will be calculated. This definition has the merits of simplicity and obviousness.

Consider the space-like components of a Riemannian space which are given by

$$ds^2 = g_{ij}\,dx^i\,dx^j,$$

where $i, j = 1, 2, 3$. In this metric there is defined at every point the invariant element of proper volume

$$dV = \sqrt{g}\; d^3x.$$

Note that despite having previously stressed the importance of the time-element, we are considering only the space-elements. But the space-elements are all we need to define our distribution. Once we have found σ, our previous work shows how the time-element depends on it.

The distribution we will consider first is defined by:

Definition. *The particles are distributed randomly per unit proper volume.* We can, of course, carry out the formal calculation of the standard deviation for this distribution in an Einstein universe. The space-like components of the metric are

$$ds^2 = \frac{dr^2}{1-(r^2/R_0^2)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

so that the element of proper volume is

$$dV = \frac{r^2 \sin\theta}{\sqrt{(1-r^2/R_0^2)}}\, dr d\theta d\phi.$$

Thus the chance of finding a particle in the range $(r, r+dr)$ would be proportional to $r^2 dr/\sqrt{(1-r^2/R_0^2)}$. No interest attaches, however, to finding the standard deviation of such a distribution, since r is not a *measured* coordinate. If we write $r = R_0 \sin\chi$, we have

$$ds^2 = R_0^2 [d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)],$$

which shows $R_0\chi$ to be the measured radial distance. In terms of χ we have

$$dV = R_0 \sin^2\chi \sin\theta\, d\chi d\theta d\phi,$$

so that the chance of finding a particle in the range $\{R_0\chi, R_0(\chi+d\chi)\}$ is proportional to

$$A \sin^2\chi\, d\chi = \tfrac{1}{2}A\, d(\chi - \sin\chi\cos\chi),$$

where A is a constant. For this to define a probability distribution we have

$$\tfrac{1}{2}A \int_{\chi=0}^{\chi=\pi} d(\chi - \sin\chi\cos\chi) = 1$$

(the limits being fixed as the values of χ for which dV vanishes), which gives $A = 2/\pi$. The second moment about the mean $(\frac{1}{2}\pi)$ is thus

$$\bar{\mu}_2 = \frac{1}{\pi} \int_{\chi=0}^{\chi=\pi} (\chi - \tfrac{1}{2}\pi)^2\, d(\chi - \sin\chi\cos\chi),$$

i.e.

$$\bar{\mu}_2 = \frac{\pi^2}{12} - \frac{1}{2},$$

i.e.

$$\bar{\mu}_2 = 0{\cdot}3225.$$

The standard deviation of χ is $\sqrt{(\bar{\mu}_2)} = 0{\cdot}568$, so that κ, which is the standard deviation of the measured length $R_0\chi$ is $0{\cdot}568R_0 = R_0/1{\cdot}761$. Hence, it follows from section 2.3, that the value of σ is $R_0/1{\cdot}761\sqrt{N}$. This is of the same order as Eddington's $R_0/2\sqrt{N}$ or as Slater's correction $R_0/3\sqrt{N}$.

Since σ and the linear scale-uncertainty, σ_ϵ, are connected by (6) we have that the value of σ_ϵ is

$$\sigma_\epsilon = \frac{1}{1{\cdot}761\sqrt{N}}.$$

However, this calculation has only a formal significance because the mean value $\frac{1}{2}\pi$ of χ is simply 'half-way round the universe', and the particles there are spread round the equator, not at one point. The centroid of the particles is equally likely to be anywhere, as is obvious geometrically. Eddington was early aware of this difficulty and in an early draft (S, C, 33, i, § 6) he deliberately goes over from an Einstein universe to another abstract structure (it is not easy to see whether a simpler or more complex one), viz. a sphere of N particles in a flat space. If the sphere is of radius R_1, so that

$$x^2+y^2+z^2 \leqslant R_1^2,$$

we have $$3\sigma_x^2 = \sigma_r^2, \quad \text{i.e.} \quad \sigma_x = \sigma_r/\sqrt{3},$$

and for one particle

$$\sigma_r^2 = \frac{3}{4\pi R_1^2}\int\limits_0^{R_1} r^4 \sin\theta\, dr d\theta d\phi,$$

so that the standard deviation for the centroid is now

$$\frac{R_1}{\sqrt{(5N)}} = \frac{R_1}{2{\cdot}236\sqrt{N}}.$$

This calculation is now unexceptionable, but the difficulty is transferred to the jump between abstract structures. (It will be found generally throughout this book that the difficulties can always be referred to this one difficult concept.) In the present instance, apart from the difficulty of knowing how it is ever possible to work in two abstract structures at once, we do not know the relation between R_0 and R_1. If we assume $R_0 = R_1$ we have a factor 2·236 which lies between Eddington's and Slater's results: but it is at least equally plausible to choose the volumes of the Einstein space and the sphere equal, so that

$$\tfrac{4}{3}\pi R_1^3 = 2\pi^2 R_0^3,$$

and then $$\sigma = \frac{R_0}{1{\cdot}333\sqrt{N}}.$$

Whichever method is adopted, Eddington argues that this scale-uncertainty is the result of measuring with a standard whose actual extension (in the geometrical coordinate frame) is uncertain to this extent. The observational measures are first represented, without correction, in a flat space. (This accounts for Eddington's use of the term 'measured distance r', although, as we remarked above, in an Einstein universe, r is not a measured coordinate.) The probability distribution in the flat space is partly affected by the fluctuation of the standard employed, and in order to obtain the true distribution corresponding to an exact standard, we must eliminate the effect of the fluctuation of the standard. This elimination changes the flat space into an Einstein universe, since by taking account of σ_ϵ, the radial part of the line-element changes from being proportional to dr/σ to being proportional to $dr/(\sigma^2-\sigma_\epsilon^2 r^2)^{\frac{1}{2}}$. This means that the curvature of the universe replaces the fluctuation of the standard, so that when we consider the particles to be in an Einstein universe, we may treat the scale of measurement of r to be exact.

Eddington remarks that the curvature representation of σ_ϵ is suitable for molar physics, but not for quantum theory, since the latter is not concerned with statistical averages of large numbers of particles. For quantum theory, he decides to revert to flat space and to take account of the scale fluctuation in another way which he describes in **FT**, § 24.

2.5. Eddington's § 4 of **FT** serves merely to discuss the validity of defining the unit of length in terms of σ. He asserts that the definition of length and the definition of time interval are required at the beginning of physics, since definitions of other physical quantities presume the existence of a system of space and time measurement. Hence, in specifying the standard referred to in the definition of length, the quantitative part of the specification must consist entirely of pure numbers. In quantum theory the physical structure is specified by pure numbers, i.e. numbers of elementary particles in configurations or states defined by quantum numbers. Because of this Eddington aserts that the standard of length must be a quantum-specified structure.

The equations of quantum theory determine the various extensions of space in quantum-specified systems as fixed multiples of a unit $\hbar/m_e c$, and the ratio of two quantum-specified extensions at the same time and the same place is a fixed constant. It follows that all quantum-specified structures give equivalent metrics which differ from one another only by a constant factor.

Later in **FT**, Eddington asserts that the Rydberg constant for hydrogen, $\mathfrak{R}$, is given by

$$\mathfrak{R} = \frac{16\pi\sqrt{5}}{3}.136^2.137.\sigma. \tag{7}$$

This means that at least one quantum-specified extension has a fixed ratio to σ, so that the unique quantum-specified metric is the same as the σ-metric. In other words, from (7), the use of the H_α line as a standard of length available at all times and places is equivalent to using the σ-metric.

The argument of Eddington's reproduced above seems fairly straightforward, although the statement that the ratio of two quantum-specified extensions at the same time and the same place is a fixed constant is not at all clear. He uses this statement later in § 4 when he points out that the ratio of the wavelength to the period of H_α light (i.e. the velocity of H_α light) is a fixed constant, so that the velocity of light is constant at all times and at all places.

In order to define length measurement in terms of more primitive ideas (as it seems Eddington wishes to do) it would be necessary to show that the working of section 2.4 could be presented in a form not involving length measurement. This would probably present some language difficulties, since it would be difficult to explain something more fundamental than length, when length is one of the fundamental ideas in physics. However, we will assume that this is possible, and that it is possible to define the unit of length in terms of σ, so that we may keep in step with the ideas that follow in **FT**.

2.6. The value of $R_0/\sqrt{N}$ is suggested, by rough estimates from astronomy, to be of the order of 10^{-13} cm, so that by considering σ we are brought into the field of nuclear theory. Eddington draws attention to the important distinction between calculated and directly measured coordinates and distances. Consider two particles very close together, such as in a nucleus, or two protons in a scattering experiment. If ξ_r, ξ_s are the physical coordinates of the two particles, then the calculated relative coordinate is the coordinate difference $\xi_{rs} = \xi_r - \xi_s$. However, we may measure the relative coordinate directly without considering the origin; this directly measured relative coordinate is called ξ'_{rs} by Eddington.

Since they are the relative coordinates of two entities both of which have uncertainty of position and momentum in the geometrical frame, ξ_{rs} and ξ'_{rs} are both observables. They have the same mean value, but their probability distributions are different. ξ_{rs} includes the coordinate differences of two random points in the distribution of the origin; a quantity with standard deviation $\sigma\sqrt{2}$. This quantity can be eliminated by measuring directly from one particle to the other, so that we have the relation

$$\xi_{rs} = \xi'_{rs} \pm \sigma\sqrt{2}.$$

Because of these two different means of defining the relative coordinates of two particles, it is necessary, when dealing with distances, to state explicitly which of the two quantities

$$r_{12} = (\xi_{12}^2+\eta_{12}^2+\zeta_{12}^2)^{\frac{1}{2}} \quad \text{and} \quad r'_{12} = (\xi_{12}'^2+\eta_{12}'^2+\zeta_{12}'^2)^{\frac{1}{2}}$$

is meant. These distinctions are unprovided for in classical terminology and they arise only through the uncertainty of the origin. Even so, the difference between r_{12} and r'_{12} is quite insignificant unless we are dealing with distances of the order of the dimensions of a nucleus. In such a case it is essential to distinguish r_{12} and r'_{12}.

Eddington then seeks to relate this to nuclear forces but this section is extremely obscure. The following slightly amended form of Eddington's argument was suggested by Bastin and Kilmister (**17**).

The Coulomb energy arises as a solution of Maxwell's equations; these are field equations and refer to measurements from an origin, so that they have a solution e^2/r. A pair of protons cannot be superposed on a neutral environment since they induce a charge in it. Some correction to the energy is necessary, but if the protons are a measured distance apart (i.e. $r' \neq 0$), the potential energy is all Coulomb energy (by a simple argument involving the effect of a reversal in sign of a charge). Hence, it follows that any correction to the energy must be a singular energy associated with actual coincidence, i.e. $r' = 0$. In order to have any effect this energy must be infinite, so if we make the assumption that the total non-Coulombian energy for a distribution of two protons anywhere in a flat space or Einstein universe is finite, then the correction will be of the form $B\delta(r')$, where δ is Dirac's δ-function, and B is a constant.

We have to express the non-Coulombian energy as a function of r. Now the origin has standard deviation σ, so that the frequency function for r is $K\exp[-(r-r')^2/4\sigma^2]$, where K is a constant. Hence, the expectation value of the non-Coulombian energy is

$$\int BK\exp[-(r-r')^2/4\sigma^2]\delta(r')\,dr' = Ae^{-r^2/4\sigma^2},$$

where A is a constant. This is one of the potentials assumed in nuclear theory, with range-constant $r_0 = 2\sigma$. (Eddington denotes the range-constant by k.)

This is a somewhat improved form of the very muddled explanation of this work given by Eddington in **FT**. However, the following four difficulties still remain:

(*a*) In scattering experiments we may do the calculations by taking the origin on the scattering proton. The use of barycentric coordinates, with its subsequent conversion (**18**, pp. 101–2), is equivalent to this, since the coordinate transformation does not involve changes of frequency functions. In this case the radial distance is measured, so that the Coulomb potential should be e^2/r'. Hence, the question of whether the Coulomb potential is e^2/r or e^2/r' cannot be settled uniquely for all cases.

(*b*) Eddington is here considering a particularly simple abstract structure since subsequent arguments presuppose energy, which is equivalent to assuming that a true potential is adequate for the description of nuclear forces. Rosenfeld (**18**, p. 314) points out that it may be necessary to consider velocity-dependent, many-body, or non-central forces. However, for the purpose of laying the foundations of the theory we may confine ourselves to very simple ideas in which only central, static, two-body forces occur.

(*c*) Rosenfeld (**18**) shows that the nuclear forces are known to be charge-independent to a high degree of approximation, but here the result for the nuclear potential is apparently charge-dependent, since the argument begins with the Coulomb potential. But Eddington preferred to analyse matter into protons and electrons. This is, presumably, theoretically possible, and it would not then be difficult to give arguments for the charge-independence of the result.

(*d*) Eddington assumes a δ-function for the non-Coulombian potential which implies the assumption that the total non-Coulombian energy is finite. This assumption is equivalent to assuming the Gaussian potential directly and is difficult to justify physically. If, for example, the function $B[\delta(r')]^2$ had been assumed for the potential, an infinite value would have resulted for the non-Coulombian term.

Eddington now attempts to provide an observational test for his theory. The mass, M_0, of an Einstein universe is given by

$$\frac{\gamma M_0}{c^2} = \tfrac{1}{2}\pi \mathrm{R}_0,$$

where γ is the gravitational constant. (Eddington denotes this by κ.) If N is the total number of particles and m the mass of the hydrogen atom, then $M_0 = \frac{1}{2}Nm$ and we have

$$\frac{\mathrm{R}_0}{N} = \frac{\gamma m}{\pi c^2} = 3{\cdot}95\times 10^{-53}\ \mathrm{cm}.$$

The experimental determination of the range-constant gives

$$r_0 = 2\sigma = \frac{2\mathrm{R}_0}{1{\cdot}761\sqrt{N}} = 1{\cdot}9\times 10^{-13}\,\mathrm{cm}.$$

Eddington uses these two equations (note that we have corrected the latter) to find the limiting speed of the recession of the galaxies, given as $V_0 = c/(R_0\sqrt{3})$ by Lemaître's formula. However, in dealing with an Einstein universe, Eddington is putting forward a very simple abstract structure; certainly one in which recession of the galaxies cannot be described, since the universe is static. If, however, we consider a universe expanding from an initial Einstein state, we find, by using the density-particle correlation (see Appendix II), that V_0 is given by

$$V_0 = \frac{c}{R_0} = \frac{4053\alpha}{\beta^2}\,\text{km/sec/megaparsec}$$

where $1{\cdot}19 < \beta < 2{\cdot}5$ and $\alpha = R/R_0$, where R is the radius of the universe at present. It is believed that R is such that $2 < \alpha < 5$ approximately. The minimum value of V_0 corresponds to the maximum value of β and the minimum value of α and is given by

$$V_0 = \frac{4053\times 2}{(2{\cdot}5)^2}\,\text{km/sec/megaparsec,}$$

so that $V_0 = 1{,}297$ km/sec/megaparsec. This is at least six times greater than the greatest observed value of the speed of recession and probably nearer to fifteen times (**19**, p. 39) and seems to be quite beyond the bounds of possibility. Thus Eddington's observational test for his determination of the nuclear range-constant is totally unsatisfactory. The only possibility, which is not a likely one, is that the clearing up of the difficulties about the jump between abstract structures might produce substantial numerical alterations. Until these difficulties are cleared up we can only conclude that it is very unlikely that Eddington's type of calculation can give results in good agreement with experiment, needing as it does a standard deviation of about $R_0/(7{\cdot}4\sqrt{N})$.

3

THE STANDARD URANOID AND SCALE-FREE PHYSICS

3.1. HAVING described the basic argument 'A' of Eddington's theory, we now turn our attention, in the next two Chapters, to argument 'B', which is to be found in paragraphs 7 to 16, 18, 19, and 22 of **FT**.

One of the outstanding features of **FT** is the importance Eddington attaches to the background in every problem. He considers that the universe may be divided into two parts, namely, an *object-system*, which is the part that is being intensively studied, and its *environment*, or background, which comprises the rest of the universe. This is simply his way of expressing the importance of hierarchies of abstract structures, as explained in Chapter 1. Eddington insists that in every problem the environment must be considered together with the object-system. In his own words, *we can no more contemplate an atom without a physical universe to put it in than we can contemplate a mountain without a planet to stand it on.* The point is that the environment is needed in order to carry out the experiments which Eddington is contemplating; and so the abstract structure must be complex enough to describe it. This does not mean that local phenomena are supposed to be more affected by the remote parts of the environment than is normally admitted by relativity theory. Many physicists regard the consideration of small-scale problems in anything but a flat-space as being a waste of time as the effect of the remote gravitational field is negligible. But Eddington's view was that though it may seem pointless to consider the remote parts of the environment, in fact it is much less troublesome to consider the whole environment, since, if we divide it, we introduce a boundary, and so give ourselves the trouble of discovering the boundary conditions. These boundary conditions would have the same effect as the continuation of the environment beyond the boundary, so it would be much simpler to consider the environment in its undivided state in each problem.

The laws of physics are initially formulated for very simple object-systems, such as electrons, two-particle systems, etc., and are later amended so that they may apply to more complex systems. In the same way as we begin by considering these simple systems, so we must first construct abstract structures by considering very simple environments, e.g. uniform, electrically neutral, etc. Eddington calls these ideally simplified environments *uranoids.*

If two problems are worked out in different uranoids we will have, as we have seen, difficulties in comparing the results, so it is advisable to take one particular uranoid as the standard uranoid which we associate with each problem. Eddington defines the standard uranoid by imposing the following conditions.

(i) It is a steady uniform probability distribution of particles, i.e. an Einstein universe.
(ii) It is at zero temperature, so that it consists of material particles only and no radiation.
(iii) It is electrically neutral.

The last condition does not imply that we cannot consider large-scale electromagnetic fields, but that we must consider such fields as being included in the object-system. The standard uranoid corresponds, not to one abstract structure, but to a certain restricted class of such structures. We have shown earlier (section 2.2) that Eddington considers a large system of particles from which he determines the uncertainty of the physical origin and hence the scale of the various physical structures. Here, the environment of the object-system is also a large assemblage of particles and we have to consider the physical interaction of the object-system with the particles of the environment. We found in section 2.2 that the particles of the universe had to be non-interacting so that we could use the Central Limit Theorem, but we assume that the particles of the object-system can interact with those of the environment.

Two different considerations have thus led us to a large assemblage of particles. These two effects are really the same, since the first is the principal geometrical effect and the second is the principal mechanical effect, and these two are both described as a 'field of $g_{\mu\nu}$' in relativity theory. The fact that particles of the standard uranoid do not interact with themselves is not a disadvantage. These particles are not like particles of an object-system, i.e. they are not particles *in* an environment, they *are* the environment. They do not provide a gravitational field amongst themselves. The gravitational field arises only when there is an object-system for the particles of the environment to interact with. This is precisely the same as in the geometrical effect, since these measurements cannot be made unless we have an object to measure.

As Eddington puts it: 'It makes no difference whether we regard the field of $g_{\mu\nu}$ as influencing the measured characteristics of the object-system by physical interaction or by determining the measure system to which they are referred.'

Eddington describes the field of $g_{\mu\nu}$ as the 'gravitational-inertial field', so that the standard uranoid provides the inertial part and the deviation of the actual environment from the standard uranoid provides the gravitational part. This explains Eddington's use of 'gravitational' fields in his development of relativistic quantum theory. No more importance is attached to the gravitational part than is usual in microscopic physics, but what is important is the inertial part, since without inertia the masses of the particles are unaccounted for. So whenever Eddington considers an object-system in a standard uranoid and uses the term 'gravitational field' he is referring to the inertial field which accounts for the masses of the particles. This differentiation may be made clearer as follows: the $g_{\mu\nu}$ field in general relativity describes that particular sort of gravitational field (considered by Newton and others) which can in virtue of Galileo's results be transformed away at any selected point. At first sight this suggests that this field cannot be given such an important physical significance as Eddington implies; rather the curvature tensor $R^{\mu}_{\nu\sigma\rho}$, through the equation of geodesic variation, determines the field separating two nearby particles. But this argument ignores something; the transformation to a coordinate-system in which the Newtonian field vanishes at a point has still to be carried out; a set of preferred coordinate-frames is determined, and this is at least part of the inertial properties of particles at the point. (Whether other inertial properties, e.g. the proper mass, are determined according to Mach's principle depends on the particular gravitational equations chosen.)

3.2. For a complete description of any system we require to know two things, namely, the internal structure of the system, which can be described wholly by numerical ratios, and the relationship between the size of the system and the rest of the universe. This relationship would be unnecessary if the universe were investigated as one system, but in analytical physics we consider only individual simple systems into which the universe is divided. The observer relates each of the systems he surveys to the others by means of an extraneous standard scale, so that the separate fragments may be combined in a definite scale-ratio to give a complete description of the universe.

It would seem that three extraneous standards, namely, length, time, and mass, would be required to complete the description of a system. But Eddington has shown earlier that if σ_ϵ is the linear scale-uncertainty, it is related to the uncertainty constant σ by

$$\sigma = \mathrm{R}_0 \sigma_\epsilon .$$

Hence, if we know the fluctuation of the standard of length, we can easily fix the scale of the whole system. Since one standard is sufficient to fix the scale, we must somehow eliminate the other two standards, or, at least, relate to σ_ϵ the standard deviations σ_m and σ_t which correspond to mass and time in the same way that σ_ϵ corresponds to length. Eddington does this by adopting a system of so-called 'natural units' which are such that

$$c = 1, \qquad 8\pi\gamma\hbar^2 = 1. \tag{8}$$

These relations are chosen because the former renders synonymous the terms mass and energy, i.e. the rest-energy $m_0 c^2$ of a particle is now m_0, the rest-mass of the particle, and the latter simplifies the relationship between the energy tensor and the momentum vector which Eddington introduces in **FT**, § 8.

Using these natural units, mass and time can be expressed in terms of length, since (8) gives:

$$L = T \quad \text{and} \quad M = L^{-3},$$

so that time has a scale-uncertainty σ_ϵ and mass has a scale-uncertainty $-3\sigma_\epsilon$, i.e. a particle would be attributed with a mass of $1 \mp 3\sigma_\epsilon$ units instead of 1 unit.

Eddington asserts that if any two other relations are chosen instead of (8), the same results will eventually be arrived at, but it is not at all obvious that this will be so. It would be better to eliminate the standards of mass and time by adopting two undefined relations:

$$f_1(L, M, T) = 0 \quad \text{and} \quad f_2(L, M, T) = 0, \tag{9}$$

between fundamental constants instead of (8). Then by developing the theory using the relations, it will be found, if Eddington is correct, that they will not appear in the final results. On the other hand, it may well be found that Eddington's results only hold good for certain definite relations, in which case there may be considerable difficulty in explaining the physical reason for this preference.

The momentum vector p_μ in quantum mechanics, defined by

$$p_\mu = -i\hbar(\partial/\partial x^\mu), \tag{10}$$

has the dimensions L^{-3} in natural units, and the energy tensor $T_{\mu\nu}$ of relativity theory, which is defined by

$$-8\pi\gamma T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G, \tag{11}$$

has dimensions L^{-6}. Because of this Eddington states that, 'An energy tensor is, both dimensionally and tensorially, the product of two momentum vectors', i.e.

$$T_{\mu\nu} = p_{\mu} p_{\nu}. \tag{12}$$

This is the relation which prompted us to develop the density-particle correlation of Appendix II. If natural units were not employed, (12) could be written in the dimensionally correct form

$$T_{\mu\nu} = D \cdot \frac{c^4}{8\pi\gamma\hbar^2} p_{\mu} p_{\nu}, \tag{13}$$

where D is some arbitrary dimensionless constant.

There are three major disadvantages to (12). Firstly, $T_{\mu\nu}$ is defined everywhere in space, whereas p_{μ} is defined only on the world-line of the particle, and secondly, the restriction placed on $T_{\mu\nu}$ that it should be the product of two vectors means that it is by no means as general as the energy tensor that is usually employed in relativity theory. But the most obvious difficulty is that p_{μ} as defined by (10) is an operator, while $T_{\mu\nu}$ is not. This difficulty is overcome in Appendix II by setting up the density-particle correlation, in which both sides of (13) operate on a scalar function ψ, operation by $T_{\mu\nu}$ being ordinary multiplication. (Alternatively, we could overcome the difficulty by assuming that p_{μ} is defined, as in relativity theory, by

$$p^{\mu} = m_0 \frac{dx^{\mu}}{ds},$$

where ds is the proper time. This is not an operator, so there is no need to treat (13) as an operator equation.) But a much more extensive discussion of (13) is really needed; see sections 4.6, 4.7.

By similar reasoning, Eddington also asserts that a particle-density, i.e. the number of particles, or probability of a particle in unit volume is a momentum vector. Both these statements are based on the assumption that if two quantities have the same physical dimensions and are represented by tensors of the same rank, then they are equivalent. In general, of course, this assumption is quite false, but it may possibly be true in these cases. It seems that the difficulties are relieved to a certain extent if we assume that the $T_{\mu\nu}$ in (13) is not the general energy tensor defined by (11), but is a particularly simple tensor, having the dimensions of an energy-tensor, suitable for the description of a system in particularly simple conditions. This seems a reasonable assumption since, at this stage, Eddington is primarily concerned with simple object-systems and simple environments.

Despite these attempts to retain Eddington's relation (12), we shall show in section 3.4 that it suffers from another serious difficulty.

3.3. The standard uranoid, or Einstein universe, which Eddington uses is characterized by two linear constants, namely, the microscopic uncertainty constant, σ, and the macroscopic radius of the universe, R_0. These are not independent, since $\sigma = R_0/1{\cdot}761\sqrt{N}$, or some similar relation, but we will regard them as being independent for the moment. Eddington then divides standard abstract structures into three, as follows:

(i) one describing scale-free physics, involving neither σ nor R_0,
(ii) one for cosmical physics, involving R_0 but not σ,
(iii) one for quantal or scale-fixed physics, involving σ but not R_0.

Scale-free physics deals with systems which are large compared with σ, but small compared with R_0. These systems can be adjusted to any scale, so long as they are not increased to the size of R_0 or reduced to the size of σ. The criterion of a scale-free system is formally stated by Eddington as follows: If we specify the characteristics of a system in terms of an extraneous standard, and consider the series of systems formed by varying the standard but keeping the specification the same, then (for a scale-free system), if one system of the series is physically possible, all are possible.

Of course, the exact equations of all the branches of physics involve both σ and R_0, but for scale-free systems a sufficiently high approximation may be obtained by putting $\sigma = 0$ and $R_0 = \infty$. Similarly, for cosmical systems, we leave R_0, but put $\sigma = 0$, and for scale-fixed systems we leave σ, but put $R_0 = \infty$.

Classical molar theory involves neither σ nor R_0, and so is contained in scale-free physics, while molar relativity theory is contained in (i) and (ii), since, although the theory introduces the concept of a finite universe, there is a considerable part of it which is not based on this idea. Similarly, quantum theory is contained in (i) and (iii). The term 'quantal theory' refers to the part in which quantization and discrete eigenstates are introduced, but there are many investigations in which the actual size of the system is not involved, e.g. in investigations to determine the mass-ratios of different kinds of particles. Hence, if some sort of unification between relativity theory and quantum theory is to be achieved, the abstract structure for scale-free physics will provide the basis for such a unification, since it is less complex than both.

3.4. In § 10 of **FT**, Eddington introduces two types of wave functions which correspond to the scale-fixed and scale-free parts of quantum theory. The wave functions used in the scale-fixed part are discrete and self-normalizing, since they refer to the part in which quantization and discrete eigenstates are introduced. They represent a particle density which rapidly decreases outwards so that the integral over space converges. We can adjust these wave functions so that they give a normalized density $s_n(x, y, z)$, whose integral over the whole domain of (x, y, z) is 1, so that s_n is said to correspond to unit occupation. If we associate an occupation factor j with the state defined by the wave function, then the corresponding density is js_n and its integral over the whole domain of (x, y, z) is j. This occupation factor gives the number of particles, or the probability that there is a particle in the state.

The scale-free or 'pseudo-discrete' wave functions are similar to the 'infinite plane waves' of elementary wave mechanics, though here we have not an infinite distribution, but one which extends uniformly to an undefined distance large compared with σ but small compared with R_0, i.e. the domain of scale-free physics. Since we cannot specify the total number of particles in a distribution of undefined extent, we choose an arbitrary normalization density s_n which is to be defined as the 'density of unit occupation'. This is equivalent to choosing an arbitrary normalization volume V_n and defining unit occupation to be one particle per volume V_n. This definition must not be taken to mean that each particle is distributed over a cell of volume V_n, since every one of the particles has a uniform probability distribution over the whole extent of the wave. In fact, a particle occupying a pseudo-discrete wave function is an unidentified member of a large assemblage.

These 'infinite plane waves' which represent uniform probability distributions are also used in more complicated systems. Eddington states that atoms, for example, are represented, not by non-uniform distribution wave functions, but by correlating the uniform distribution wave functions of the electrons and the nuclei. He gives the following example to illustrate this point. Consider a vessel known to contain a proton and an electron, each of which is equally likely to be anywhere in the vessel. In due time they will combine into a hydrogen atom emitting a photon. The electron is still equally likely to be anywhere in the vessel; and so also is the proton. Their probability distributions, while remaining uniform, have become correlated. Eddington calls such atomic wave functions, 'correlation wave functions'. So we can see that uniform distribution wave functions are used not only to represent simple systems, but also to provide a starting-point for considering more complicated systems. Only in a few special problems, such as the deflexion of particles in a molar electromagnetic field, are we required to use non-uniform distribution wave functions.

Eddington uses the term 'pseudo-discrete' instead of 'continuous' to describe scale-free wave functions, because, he asserts, they cannot be obtained by replacing a set of discrete distribution functions $f_\alpha(x, y, z)$ by a continuous distribution function $f(x, y, z, \alpha)$ when α becomes continuous. The reason Eddington gives for the breakdown of this method in this case is that wave functions contain a phase which gives rise to interference when two or more waves exist at the same point, and it is evident that a representation in which waves at the 'point' (x, y, z, α) interfere is not the limiting form, for continuous α, of the representation in which waves with different values of α at the same point (x, y, z) interfere. Eddington's solution to the problem is to divide the domain of α into arbitrary small ranges $\delta\alpha_r$ and associate a wave function ψ_r with each range. These ψ_r are the pseudo-discrete wave functions.

The term 'almost exact' is associated with pseudo-discrete states and is used extensively in later work where a system in a state of almost exact rest is often considered. It is defined as follows:

When the whole occupation is concentrated in a range represented by one pseudo-discrete wave function, the state of the system is said to be *almost exact*.

Eddington then considers the two different types of particles which are described by scale-free and scale-fixed wave functions. He first deduces that mass is a scale-fixed characteristic, whereas density is a scale-free characteristic. The argument involved is somewhat obscure, but the conclusion can be explained in the following way: The fluctuation of the standard of mass is connected, by the relations (9), to the linear scale-uncertainty σ_ϵ, which, in turn, is connected to the scale. If the scale is doubled, say, then the mass is changed correspondingly. Hence, the mass depends on σ, and so is a scale-fixed characteristic. The same does not apply to the density since the proper density may be varied at will by varying the volume over which its probability distribution extends, so that whatever changes take place in the scale, the proper density can always be kept at the same value. Hence, the density is independent of σ, and so is a scale-free characteristic. Since proper density is represented as the component T_{44} of the energy tensor $T_{\mu\nu}$ and the proper mass as the fourth component of the momentum vector of the particle, we can express the above result in the form: the particles of scale-free physics are carriers of energy tensors, and the particles of scale-fixed physics are carriers of momentum vectors.

These results make Eddington's relation

$$T_{\mu\nu} = p_\mu p_\nu$$

seem even more curious, since here he is relating a scale-free characteristic, $T_{\mu\nu}$, to a scale-fixed characteristic, p_μ. This relation appears meaningless, since the two sides of the equation refer to completely different situations. (See, however, section 4.7.)

Although the masses of particles are scale-fixed, the ratios of masses can often be expressed in terms of density. For example, the mass-ratio m_p/m_e of a proton and electron can be expressed as the density-ratio of the positive and negative constituents of a molar distribution of hydrogen, and so it comes within scale-free theory.

We have started with a standard uranoid which consists of a large assemblage of particles, and is characterized by σ and R_0. Scale-free physics is that branch of physics which is concerned with distributions of undefined extent which are large compared with σ, but small compared with R_0. We still have a large assemblage of particles in such a distribution, but the individual particles (scale-free particles) are unidentified. A molar object, i.e. one which is large compared with σ and small compared with R_0, can be represented by means of pseudo-discrete wave functions, as a large assemblage of scale-free particles in one or more pseudo-discrete states. Each particle is a carrier of an energy tensor $\Delta T_{\mu\nu}$, and so the molar energy-tensor is the sum of the contributions $\Delta T_{\mu\nu}$ from each particle.

Since the individual particles are unidentified members of the whole assemblage, we know only the probability of any one particle being in the various states, so that the occupation factors j_r of the pseudo-discrete states can be interpreted as the probabilities of the particles as well as the frequencies in the assemblage. Now the probability of an individual is defined as the frequency in an ensemble, but the ensemble is usually taken to be an imagined infinite class of individuals or events, such as the imagined infinite repetition of measurements. But, in this case, the ensemble is a finite, though undefined, assemblage which is actually present in the (idealized) physical universe. The finiteness of the assemblage ought to be represented by an extraordinary fluctuation, but the scale-free approximation treats this as being negligible.

Later in **FT** (§ 33), Eddington shows that Planck's constant and the electric charge are connected by the relation $\hbar = 137e^2/c$, and asserts that, since $\hbar$ arises in quantization (which belongs to the scale-fixed part of physics), then the latter is a distinctively electrical phenomenon. The term 'distinctively electrical' is used, since we, up to a point, represent electrical characteristics by an energy tensor which can be absorbed into the mechanical energy tensor. Distinctively electrical theory starts when this procedure breaks down and the energy tensor, which is the scale-free characteristic, is replaced by the momentum vector, which is the scale-fixed characteristic, related to e^2/c or $\hbar$ (since $p_\mu = -i\hbar(\partial/\partial x^\mu)$). So scale-free theory may be described as mechanical theory, since it is associated with the energy tensor and scale-fixed theory may be described as electrical theory, though some parts of electrical theory may be transferred to scale-free theory.

3.5. A pure observer must measure all the physical characteristics of the object-system that he is considering. Any observational measurements that he makes cause a reciprocal uncertainty in the conjugate variates. But an actual observer steals 'free information' about the system he is studying, e.g. if he says he is studying an electron, he will use the tabulated values of m and e instead of his own rougher measurements. This free information is not paid for by reciprocal uncertainty in the conjugate variates. Eddington calls a quantity of an abstract structure whose value is given as free information a *stabilized characteristic*. A stabilized characteristic is thus a quantity whose value is prescribed as one of the postulated conditions of a theoretical problem, and so is not an observable. The whole standard uranoid is a stabilized characteristic, since when we consider some object-system in a standard uranoid, we are postulating that the environment of the system is uniform, static, of zero temperature and electrically neutral.

Eddington considers the problem of a particle of proper mass m with momentum 4-vector (p_1, p_2, p_3, p_4). The first three components p_1, p_2, p_3 are directly measured, i.e. they are observables, but he asserts that p_4, which he calls the *hamiltonian*, is usually considered to be computed by the formula $p_4^2 = p_1^2+p_2^2+p_3^2+m^2$ $(c = 1)$. Two very important cases now arise, depending on whether we consider m to be an observable like p_1, p_2, p_3, or a stabilized characteristic of the particle. If m is an observable, then the probability distribution of the mechanical characteristics of the particle is a four-dimensional one, either over p_1, p_2, p_3, m or, equivalently, over p_1, p_2, p_3, p_4. However, if m is a stabilized characteristic then it does not enter into the probability distribution, so the latter is reduced to a three-dimensional distribution. The domain of the probability distribution is called the phase space by Eddington and if it has k dimensions the particle is called a V_k. Hence, a simple particle, i.e. one without spin, is a V_3 or a V_4 according as its proper mass is or is not stabilized. Stabilization corresponds to introducing constraints which reduce the number, k, of degrees of freedom of the system, or equivalently, of the abstract structure. If k is changed we are altering the complexity of the abstract structure, and so altering the values of some properties of the system. It is found that the number k is of considerable importance in later developments of **FT**.

Stabilization of the individual components of a vector or tensor destroys its tensor transformation properties. However, we may stabilize a tensor by imposing *on the tensor as a whole* certain invariant conditions which it must satisfy, such as the condition that a second-rank tensor is antisymmetric, or the outer product of two vectors, or the outer square of a vector. A tensor subjected to these conditions still satisfies the tensor transformation properties, but the number of independent variates required to specify it is reduced, so that the number of dimensions of its probability distribution is also reduced.

3.6. When we consider an object-system, we must also consider its environment, which we usually take as a standard uranoid. But the object-system will disturb the standard uranoid by gravitational and electromagnetic interaction, so, in any problem, the environment is not as simple as the standard uranoid, but can be taken as the 'sum' of a uniform environment and a 'disturbance'. In terms of the algebraic picture of Chapter 1, if the object system and environment have abstract structures with k_1, k_2 degrees of freedom, the combined system will have the direct product structure of order $k_1 k_2$. In section 2.2 we required the particles of the large assemblage of particles to be non-interacting. Here we will assume that while there is no interaction between the particles of the environment, these particles do interact with those particles which are selected to form the object-system.

The universe then consists of the object-system (O.S.) and disturbed environment (D.E.) which can be subdivided into the undisturbed environment (U.E.) and the disturbance (D.). This can be written in the symbolic form:

$$\text{O.S.}+\text{D.E.} = \text{O.S.}+\text{U.E.}+\text{D.}$$

In general relativity, the disturbance is grouped with the environment, so that we have to consider a particle in a complex environment. For example, the introduction of a particle into an empty flat universe changes the latter to a Schwarzschild universe. But Eddington asserts that the method of wave-mechanics requires us to include the disturbance in the object-system. This is because the environment in wave-mechanical problems remains uniform, since the tensor $g_{\mu\nu}$ which appears in the equations remains constant no matter what changes take place in the system. When the disturbance is included in the object-system it is called the *object-field.*

We have said in section 3.1 that the environment comprises everything that is not specifically in the object-system, but in practice we specify the environment as a standard uranoid, and include everything else in the object-system. With the exception of the inertial field, which corresponds to the undisturbed uranoid, all fields are included as object-fields. These object-fields are divided into two categories, namely, *extraneous* and *complementary* object-fields. Extraneous fields are those which are deliberately introduced so that we may be able to study the behaviour of the object-particles in more complex conditions than those provided by the standard uranoid. They are, in fact, special devices for constructing new abstract structures with just the *desired* kind of increased complexity. The complementary object-field is the substitute that we put into the equations of the system to compensate for neglecting the effects of the disturbance on the environment. Instead of considering a problem to be that of an object-system in a complex environment, we transfer the disturbance to the object-system, call it the complementary object-field (provided that there are no extraneous fields), and treat the problem as one of an object-system and its object field in a standard uranoid, i.e.

$$\text{universe} = \text{S.U.} + \text{O.S.} + \text{C.F.}$$

The complementary object-field is then a measure of an inevitable increase of complexity in the abstract structure. In the symbolic equations we have used above, it is not clear what we mean by the addition signs. Eddington asserts that these signs refer to certain quantitative characteristics, so, for example, the difference between the energy tensors of the disturbed and the undisturbed environments is transferred to the object-system as the complementary field energy tensor. It follows that an object-field is specified by the same set of variates, such as energy, momentum, etc., as a distribution of particles, so that each object-particle has a particle energy tensor and a complementary field energy tensor.

Eddington associates the term 'field' with averaged characteristics, which is to be expected since he is concerned throughout with a large assemblage of particles. The 'field' energy of a particle is the interaction energy of an object-particle with the environment, i.e. it is the complementary field energy of an object-particle. Since both the disturbed and the undisturbed environments are composed of a very large number of particles, it follows that the energy of the disturbance, i.e. the field energy of the particle, is associated with averaged characteristics.

The importance of both the complementary gravitational field and the complementary electromagnetic field are stressed by Eddington who remarks that quantum theory ignores them altogether. He asserts that it makes up for the omission by empirical factors and terms in its formulae, since it is primarily an empirical theory, but gives no example to illustrate this.

3.7. Eddington points out the error of taking the background field in quantum theory to be the same as that in general relativity. In the former theory, the $g_{\mu\nu}$ which occur in the wave equation remain unchanged whatever alterations take place in the system under consideration. But in general relativity, alterations in the system are reflected by changes in the $g_{\mu\nu}$. He considers that in wave mechanics we contemplate a rigid frame of eigenstates unaffected by changes of the system, whereas in relativity theory any change of the occupying matter involves a readjustment of the field. This readjustment of the field makes relativity calculations so difficult that the relativistic method is quite useless in solving the problems with which quantum theory is concerned. For this reason Eddington elects to follow the rigid-field method of wave mechanics. This is only an approximate method, and to ensure that it will be a valid first approximation, Eddington formulates his 'rigid-field convention' in the form:

> *The field must be stationary for small changes in the occupation factors of the eigenstates.*

It is far from clear what is meant by 'must be' in this statement. We shall see later that if the field is to be stationary for *arbitrary* occupations (as Eddington seems to imply) the convention leads only to triviality. If we try to avoid this by supposing that the convention determines the occupations, as a minimal principle, it is hard to see how to fit this determination in with the other determinations of whatever particular problem is being considered.

Distinction is drawn between 'quantum particles', which are entities supposed to occupy, or to have a probability of occupying, states in a rigid field, and 'relativity particles' which are defined as singularities in an essentially non-rigid field. Eddington draws attention to the error of applying to quantum particles formulae that have been developed for relativity particles. For example, he states: 'The fact that relativity particles have Lorentz-invariant properties gives no prima facie reason for supposing that quantum particles have Lorentz-invariant properties.' Eddington's arguments against the insistence on making the equations of wave mechanics Lorentz-invariant are given in two papers he wrote (**13, 14**).

It may seem strange that the particles of the two theories are not the same, but we recognize particles only by their behaviour in certain conditions. A particle's behaviour in a rigid field will be different from its behaviour in a non-rigid field, so we regard the two cases as being of different particles. Eddington says more about this idea of 'theoretical particles' in **FT**, § 17.

4

PARTITION OF THE ENERGY

4.1. EDDINGTON continues his argument 'B' by considering the partition of the total energy between the field and the particle. Such an investigation is extremely interesting as being a first serious attempt to come to grips with the problem of the quantitative changes involved in a change of abstract structure. We shall therefore deal with it at some length, although as an argument it has serious defects. It is not clear what Eddington means by the total, field, and particle energies, but presumably particle energy refers to the particles of the object-system, field energy to the complementary object-field, while the total energy is the sum of these two, i.e. it is the energy of the object-system and its object-field but it does not refer to the particles of the standard uranoid.

In **FT**, § 14, this energy partition is considered in the case when the eigenstates are discrete. In this we have a set of states ψ_r $(r = 1, 2, \ldots)$ to each of which there is assigned an occupation factor j_r. These j_r can be regarded as generalized coordinates (or generalized momenta), since changes of the system are specified by changes of the j_r. The total energy of the system, including the complementary field, will be a function $H^0(j_1, j_2, \ldots)$ of the j_r, and this will not in general be a linear function since the complementary field has to be recomputed and the eigenstates redetermined after any change of occupation.

Let

$$E_r = \frac{\partial H^0}{\partial j_r}, \tag{14}$$

so that

$$dH^0 = E_r \, dj_r. \tag{15}$$

Since the field is rigid, the whole change of energy must be accounted for by the change of particle energy alone. Hence E_r is the energy of a particle in the state ψ_r, since (15) expresses the fact that $E_1 \, dj_1$ (say) is the change due to the addition of the fraction dj_1 of a particle of energy E_1.

The total particle energy of the system is given by

$$E^0 = j_r E_r = j_r \frac{\partial H^0}{\partial j_r}, \tag{16}$$

and the field energy is the difference

$$W^0 = H^0 - E^0 = H^0 - j_r \frac{\partial H^0}{\partial j_r}. \tag{17}$$

Eddington states that if H^0 is a homogeneous function of the nth degree in the j_r, then we have

$$E^0 = nH^0, \qquad W^0 = (1-n)H^0.$$

However, we will show that H^0 is a homogeneous function of degree one if we assume, as Eddington appears to do, that the rigid-field convention is to apply for arbitrary occupations. Since the field is rigid we have

$$dW^0 = 0,$$

for small changes in the occupation factors. By (17) this gives:

$$\frac{\partial}{\partial j_s}\left(H^0 - j_r \frac{\partial H^0}{\partial j_r}\right) dj_s = 0.$$

This is true for small changes in all or any of the occupation factors, so that for all s,

$$\frac{\partial}{\partial j_s}\left(H^0 - j_r \frac{\partial H^0}{\partial j_r}\right) = 0,$$

i.e.

$$j_r \frac{\partial H^0}{\partial j_r} = H^0 - K, \tag{18}$$

where K is a constant. From (16) this becomes

$$E^0 = H^0 - K, \tag{19}$$

so that

$$W^0 = K.$$

Hence, the field energy is a constant for *all* changes in the j_r. It follows that for discrete eigenstates the field is rigid for changes of any magnitude in the occupation factors.

By using (19), we may write (18) in the form

$$j_r \frac{\partial E^0}{\partial j_r} = E^0. \tag{20}$$

The general integral of this equation is

$$\phi\left(\frac{j_1}{E^0}, \frac{j_2}{E^0}, \frac{j_3}{E^0}, \ldots\right) = 0, \tag{21}$$

and if E^0 is a homogeneous function of the j_r, (20) shows that it must be of degree one. In this case H^0 $(= E^0 + K)$ will be a homogeneous function only if $K = 0$ (i.e. the field energy is zero), and then it will be a homogeneous function of degree one. However, there is no reason why E^0 or H^0 should be homogeneous functions; the integral (21) gives an infinite number of functions satisfying the required conditions.

Hence, on applying the rigid-field convention to a scale-fixed system, i.e. one with discrete eigenstates and discrete occupation factors, j_r, we find, contrary to Eddington, that the field is unconditionally rigid for all changes of the occupation factors. That is, in scale-fixed physics the partition of the energy is trivial and the scale-fixed part of current quantum theory is quite justified in ignoring the effect of the field.

4.2. The foregoing treatment is now applied by Eddington to scale-free systems. Since the eigenstates are no longer discrete, we cannot specify them by sets of quantum numbers in the usual way. Instead we let X^α ($\alpha = 1, 2, \ldots, n$) be the characteristics used for classifying the states of a scale-free system, where the X^α all have the same physical dimensions. For convenience, we adopt an extraneous standard (see section 3.2) of the same dimensions as X^α; so that X^α has dimension-index 1. This means that since the system is scale-free, we can obtain another physically possible system by the transformation $X^\alpha \rightarrow \lambda X^\alpha$, where λ is some parameter. The corresponding transformation for a characteristic Y of dimension-index l is $Y \rightarrow \lambda^l Y$.

We may picture the X^α as the coordinates of a point in an n-dimensional 'representation space', and since the nature of the system contemplated will usually be defined by a set of relations between the X^α, it follows that the possible states will form a k-dimensional locus in the representation space, where usually $k < n$. This locus is called the *phase space* by Eddington and the number k is called the *multiplicity factor*.

The metric adopted for the phase must be scale-true, i.e. one in which the scale-transformation $X^\alpha \rightarrow \lambda X^\alpha$ transforms the volume element dV into $\lambda^k dV$. (We may as well adopt a Cartesian metric for the phase space, since a representation by any other type of coordinate system leads to difficulties with the scale-transformation $X^\alpha \rightarrow \lambda X^\alpha$. Eddington seems to have been unaware of this fact.) The discrete occupation factors of the previous section are replaced by a continuous occupation function $j(X^\alpha)$ such that $j(X^\alpha)\,dV$ is the collective occupation of the states compressed in the range dV. Eddington is here really considering the continuous case and not the pseudo-discrete case, but he later asserts that by dividing the phase space into numbered cells $d\tau_r$ we replace the continuum of states by pseudo-discrete states with occupation factors $j_r = j(X^\alpha)\,d\tau_r$. This process is probably allowable; in any case, we shall ensure that the systems we study are scale-free by using the scale-free conditions in our derivation of the partition equations.

The coordinates Y^α, say, of the phase space may be regarded as the effective classifying characteristics of the states, since the Y^α represent the possible states. Thus the occupation function j may be taken to be effectively a function of the Y^α. For example, suppose that the X^α can be represented as the Cartesian coordinates (x, y, z) of a three-dimensional space. Suppose also that the possible states are defined as lying on the sphere $r = a$, so that the coordinates Y^α are (θ, ϕ), where (r, θ, ϕ) are the polar coordinates correspondlng to (x, y, z). The j is a function of the (x, y, z), but any change in j must leave the system in a possible state, i.e. a state described by the coordinates (θ, ϕ). Hence, we may consider j to be a function of (θ, ϕ) only, since if we consider j to be a function of the (x, y, z), changes in these coordinates may lead us to impossible states.

The classifying characteristics may be many things. For example, the hydrogen atom may be described by its energy, and one of the coordinates of the phase space could represent this energy. Eddington's explanation of these ideas in **FT** is not clear, but it is hoped that the explanation given here with the introduction of the Y^α clarifies this point. Now Eddington's discussion must be viewed as inviting us to consider systems in which properties like the energy depend not only on the occupations $j(Y^\alpha)$ of the states, but on the derivatives $j_\alpha = \partial j/\partial X^\alpha$ as well. The general idea is easy to grasp—the energy depends not only on how many particles are in each state, but how violently neighbouring states differ in occupation—an entropy-like consideration. But a precise physical example is still wanting.

Defining E as the energy of the particle in the states in dV and K^0 as the total energy of that part of the system in these states, then we have the following equations corresponding to those for the discrete case, i.e. (14)–(17),

$$E = \frac{\delta K^0}{\delta j}, \tag{22}$$

$$H^0 = \int K^0\, dV, \tag{23}$$

$$\delta H^0 = \delta \int K^0\, dV = \int \frac{\delta K^0}{\delta j} \delta j\, dV, \tag{24}$$

$$E^0 = \int Ej\, dV = \int \frac{\delta K^0}{\delta j} j\, dV, \tag{25}$$

$$W^0 = H^0 - E^0 = \int \left\{K^0 - j\frac{\delta K^0}{\delta j}\right\} dV, \tag{26}$$

where $\delta K^0/\delta j$ is the Hamilton derivative of K^0 with respect to j.

Eddington does not introduce K^0; instead he writes H^0 in place of K^0 in (23)–(26). This is probably an error of writing, since the reference he gives (**11**, § 60) necessitates the introduction of K^0.

Note that dV is the element of proper volume and may be written in the form $\sqrt{g}\,d\tau$, where g is the determinant of the fundamental tensor of the phase space and $d\tau$ is the element of coordinate volume.

Eddington now considers that H^0 has dimension-index l, so that from the above equations E^0, E, W^0 have all dimension-index l, and K^0 has dimension-index $l-k$. Since the system is scale-free, we can take for the variation δ an infinitesimal scale-variation $X^\alpha \to X^\alpha(1+\epsilon)$. Then

$$W^0 \to W^0(1+\epsilon)^l \quad \text{and} \quad E\,dV \to E\,dV(1+\epsilon)^{l+k},$$

so that $\quad \delta W^0 = l\epsilon W^0 \quad$ and $\quad \delta(E\,dV) = (l+k)\epsilon E\,dV.$

From (25) and (26) this leads to

$$W^0 = -\left(\frac{l+k}{l}\right)E^0, \qquad H^0 = -\frac{k}{l}E^0. \tag{27}$$

For a scale-free assemblage characterized solely by an energy tensor $T_{\mu\nu}$, we have that $T_{\mu\nu}$ (or H^0) $\equiv X^\alpha$, so that $l = 1$, and (27) becomes

$$W^0 = -(1+k)E^0, \qquad H^0 = -kE^0. \tag{28}$$

However, Eddington's derivation of these results is wrong. The variation expressed in (24) which leads to the Hamiltonian derivative, does not involve a variation of dV and so cannot be taken as a scale variation, since the volume element is transformed by a scale variation, i.e. $dV \to (1+\epsilon)^k\,dV$.

One of the authors (**20**) has obtained relations similar to (27) from considerations of observable quantities. The argument employed is rather long and complex, so we will describe here only the last few steps which are relevant to the present discussion. The equations (22)–(26) are established and consideration is then given to the scale-transformation

$$\sigma \to \frac{\sigma}{1+\epsilon}. \tag{29}$$

This changes the volume element of the space as well as the characteristics X^α, so that

$$\delta H^0 = \int \frac{\delta K^0}{\delta j}\delta j\,dV + \int K^0 \delta(dV).$$

Since $dV \to (1+\epsilon)^k\,dV$, this gives

$$\delta H^0 = \int \frac{\delta K^0}{\delta j}\delta j\,dV + k\epsilon H^0. \tag{30}$$

The condition is imposed that the observable is scale-free in a special way called reducible, which leads to

$$\delta j = -k\epsilon j,$$

so that (30) becomes

$$\delta H^0 = k\epsilon(H^0 - E^0). \tag{31}$$

On the other hand, we have that, for some $\sigma = \sigma_0$,

$$\left(\frac{\delta K^0}{\epsilon K^0}\right)_{\sigma=\sigma_0} = l(\sigma_0) \quad \text{(for some } l\text{)},$$

i.e.

$$\delta K^0 = l\epsilon K^0.$$

Hence (30) becomes

$$\delta H^0 = \epsilon \int lK^0\,dV + k\epsilon H^0. \tag{32}$$

Now H^0 is called a 'Hamiltonian property' when l is a constant, so that (32) becomes for such properties

$$\delta H^0 = \epsilon(l+k)H^0. \tag{33}$$

From (31) and (33) we have

$$H^0 = -\frac{k}{l}E^0,$$

as in (27).

In fact, the choice of l as a constant is a rather special case and one that is inconsistent with the rigid field, as we will show later. In general, from (29)

$$\epsilon = -\frac{\delta\sigma}{\sigma},$$

so that

$$l = -\frac{\sigma}{K^0}\frac{\delta K^0}{\delta\sigma}.$$

Thus, in general, l is a function of σ and K^0 and so cannot be taken out of the integral sign in (32). However, we may define a function L, such that

$$LH^0 = \int lK^0\,dV,$$

i.e.

$$L = \frac{\int lK^0\,dV}{\int K^0\,dV}, \tag{34}$$

so that L is a sort of 'mean value' of l. L is a *functional* of K^0; it depends on the range of integration, the functional form of K^0, and the scale σ. In fact, we neglect the dependence on σ, since scale-free physics results from the approximation $\sigma = 0$. Equation (32) now becomes

$$\delta H^0 = \epsilon(L+k)H^0,$$

and this leads to
$$H^0 = -\frac{k}{L}E^0, \tag{35}$$

and
$$W^0 = -\left(\frac{k}{L}+1\right)E^0. \tag{36}$$

Now if we take l to be a constant, we have, from (34), $L = l$, so that (35) and (36) lead to
$$\delta W^0 = -\left(\frac{k}{l}+1\right)\delta E^0,$$

and
$$\delta H^0 = -\frac{k}{l}\delta E^0.$$

The condition imposed by the rigid-field convention is $\delta W^0 = 0$, which, from the last two equations, leads to $\delta E^0 = \delta H^0 = 0$. Hence, if the total and particle energies are constantly related, the imposition of the rigid-field convention leads to these energies being rigid as well as the field energy. This is the reason why we have considered the more general case in which l is not a constant.

From (26) and the rigid-field convention, we have
$$\delta W^0 = \delta \int \left\{K^0 - j\frac{\delta K^0}{\delta j}\right\}\sqrt{g}\, d\tau = 0,$$

where the integral is taken over all space (or some fixed region). We then have
$$\int \frac{\delta}{\delta j}\left\{K^0 - j\frac{\delta K^0}{\delta j}\right\}\delta j\sqrt{g}\, d\tau = 0,$$

and since this has to be true for all arbitrary variations δj it follows that
$$\frac{\delta}{\delta j}\left\{K^0 - j\frac{\delta K^0}{\delta j}\right\} = 0. \tag{37}$$

Hence we have
$$K^0 - j\frac{\delta K^0}{\delta j} = a(j, j_\alpha, \ldots), \tag{38}$$

where $\delta a/\delta j = 0$, and the j_α are the partial derivatives of the j with respect to the X^α. From (26) and (38) we have
$$W^0 = \int a\sqrt{g}\, d\tau = \int a\, dV. \tag{39}$$

From (35) and (36) we have
$$W^0 = \left(\frac{L}{k}+1\right)H^0,$$

so that, applying the rigid-field convention, we have
$$\delta\left\{\left(\frac{L}{k}+1\right)H^0\right\} = 0,$$

i.e.
$$\delta\left\{\left(\frac{L}{k}+1\right)\int K^0\sqrt{g}\,d\tau\right\} = 0,$$

i.e.
$$\delta\left\{\int\left(\frac{l}{k}+1\right)K^0\sqrt{g}\,d\tau\right\} = 0,$$

i.e.
$$\int\frac{\delta}{\delta j}\left[\left(\frac{l}{k}+1\right)K^0\right]\sqrt{g}\,\delta j\,d\tau = 0,$$

and since this is true for all arbitrary variations δj, it follows that

$$\frac{\delta}{\delta j}\left\{\left(\frac{l}{k}+1\right)K^0\right\} = 0.$$

Hence, we have
$$\left(\frac{l}{k}+1\right)K^0 = b(j,j_\alpha,...) \tag{40}$$

where $\delta b/\delta j = 0$. From (26) and (40) we have

$$W^0 = \int b\sqrt{g}\,d\tau = \int b\,dV. \tag{41}$$

It does not follow from (39) and (41) that a and b are necessarily equal. We have the two relations

$$\int (a-b)\,dV = 0,$$

and
$$\frac{\delta a}{\delta j} = 0,$$

the relation $\delta b/\delta j = 0$ being contained in these two.

Since the simplest kind of scale-free particle is characterized by an energy tensor and nothing else, it follows that in this case (35) and (36) give the partition of the total energy tensor $T_{\mu\nu}$ into the particle energy tensor $E_{\mu\nu}$ and the field energy tensor $W_{\mu\nu}$. When we reduce k by applying stabilizing conditions the total energy remains unchanged, but the partition between the particles and the field is altered.

Note that from (40) it appears that l may depend on k, so that L may depend on k. If this is the case, then the variation of the partition equations (35) and (36) with stabilization is not obvious.

If we put $\theta = k/L$, then, in the case of energy tensors, the partition equations become

$$T_{\mu\nu} = -\theta E_{\mu\nu}, \qquad W_{\mu\nu} = -(\theta+1)E_{\mu\nu}, \tag{42}$$

and when the field is rigid, we have

$$\delta W_{\mu\nu} = -\delta\{(\theta+1)E_{\mu\nu}\} = 0,$$

i.e.
$$\delta\theta\,.\,E_{\mu\nu}+(\theta+1)\delta E_{\mu\nu} = 0.$$

This can be rewritten in the form

$$\delta E_{\mu\nu} = -\delta\theta . E_{\mu\nu} - \theta\,\delta E_{\mu\nu},$$

i.e.
$$\delta E_{\mu\nu} = -\delta(\theta E_{\mu\nu}),$$

i.e.
$$\delta E_{\mu\nu} = \delta T_{\mu\nu},$$

which is the expected result, since the field is rigid.

If K^0 is a function of j only, and not its derivatives, then the Hamiltonian derivatives in (37) are replaced by ordinary derivatives and we have

$$K^0 - j\frac{dK^0}{dj} = a,$$

where a is a function of X^α only. In fact, since we may regard j as a function of the Y^α only, the a may also be regarded as a function of Y^α only. This leads to

$$W^0 = \int a\,dV,$$

so that W^0 is independent of j. Hence, if K^0 is a function of j only, then the separation of the total energy into particle and field energy is merely the orthodox one, since the total energy is the particle energy plus a term independent of j, i.e. a term which remains unchanged for changes of the occupation factors, as in the discrete case.

We now choose an intial state and partition the initial energy $(T_{\mu\nu})_0$ into $(E_{\mu\nu})_0$ and $(W_{\mu\nu})_0$ by (42). Eddington calls the energy

$$(T_{\mu\nu})_0 = -\theta(E_{\mu\nu})_0,$$

the *initial energy*. Since the field is rigid, we have

$$\delta T_{\mu\nu} = \delta E_{\mu\nu} \tag{43}$$

for a small change of occupation and this Eddington calls the *transition energy*.

If we have an initial state and make a small change in the occupation factors, then we would expect the particle energy to be given by

$$E_{\mu\nu} = (E_{\mu\nu})_0 + \delta E_{\mu\nu}. \tag{44}$$

From (43) we have
$$E_{\mu\nu} = (E_{\mu\nu})_0 + \delta T_{\mu\nu},$$

which, from (42), gives

$$E_{\mu\nu} = (E_{\mu\nu})_0 - \delta(\theta E_{\mu\nu}),$$

i.e.
$$E_{\mu\nu} = (E_{\mu\nu})_0 - \theta\,\delta E_{\mu\nu} - (E_{\mu\nu})_0\,\delta\theta. \tag{45}$$

Since Eddington takes $\theta = k$, he finds the relation

$$E_{\mu\nu} = (E_{\mu\nu})_0 - k\,\delta E_{\mu\nu}, \tag{46}$$

and concludes that the rigid-field convention leads to the transition energy being $-k\,\delta E_{\mu\nu}$ instead of the expected $\delta E_{\mu\nu}$, i.e. the transition energy is $-k$ times the expected amount. However, this is an illusion since we have shown that when θ is a constant $\delta E_{\mu\nu} = 0 = \delta T_{\mu\nu}$ so that both (44) and (45) become

$$E_{\mu\nu} = (E_{\mu\nu})_0.$$

Since we are not considering θ to be a constant, we have

$$E_{\mu\nu} = (E_{\mu\nu})_0 + \delta E_{\mu\nu},$$

i.e. the transition energy has, in fact, its expected value.

The case of classical quantum theory is given by $\theta = -1$ ($k = -1$ in Eddington's notation), since this leads to $W^0 = 0$, i.e. the total energy is the particle energy only. If we put $k = -1$ in Eddington's erroneous equation (46) we obtain (44). Eddington asserts that the $E_{\mu\nu}$ in (44) is the energy tensor of a relativity particle and the transition energy expressed in (46) arises through the imposition of the quantum condition, namely, the rigid-field convention. In fact (44) is the energy tensor of any particle in a rigid field and, in particular, the energy tensor of the classical quantum particle is of this form. From this result (46) Eddington states that a relativity particle is formally a quantum particle of multiplicity -1, but this is clearly wrong; in Eddington's results it is the classical quantum particle that is given by $k = -1$.

The classical case cannot be obtained as a special case of Eddington's work since k must be a positive integer. However, with our results (42), the classical case is given by $\theta = -1$, which means that

$$k = -L.$$

Thus, in general, it is possible to arrive at the classical case from his theory, since L may be a negative integer.

We may write (44) as

$$E_{\mu\nu} = -\frac{(T_{\mu\nu})_0}{\theta_0} + \delta T_{\mu\nu}, \tag{47}$$

where θ_0 is the value of θ before a change of occupation takes place. For the classical case $\theta = -1$, so that

$$E_{\mu\nu} = (T_{\mu\nu})_0 + \delta T_{\mu\nu} = T_{\mu\nu}.$$

Now an alteration in k induces an alteration in L and θ (not to be confused with an alteration due to a change in occupation), but the total energy $T_{\mu\nu}$ remains unaltered. Also the total initial energy $(T_{\mu\nu})_0$ remains unaltered, so that the total transition energy remains unaltered. What does alter is the partition of $T_{\mu\nu}$ into $E_{\mu\nu}$ and $W_{\mu\nu}$. Hence the

$(T_{\mu\nu})_0$ and $\delta T_{\mu\nu}$ in (47) are those of the classical case and it is the $E_{\mu\nu}$ that is different. Hence, the effect of any change in θ is to multiply the expected (i.e. classical) initial particle energy by $-1/\theta_0$, but this has no effect on the transition energy. For example, corresponding to the case $k = 136$ say, the particle energy is

$$E_{\mu\nu} = -\frac{(T_{\mu\nu})_0}{{}_{136}\theta_0}+\delta T_{\mu\nu},$$

i.e.
$$E_{\mu\nu} = -\frac{L_{136}}{136}(T_{\mu\nu})_0+\delta T_{\mu\nu},$$

where ${}_{136}\theta_0$, L_{136} are the values of θ_0, L corresponding to $k = 136$.

The results of this section (**FT**, § 15) are very important to Eddington's theory, but are incorrect. Eddington's partition equations (28) should be replaced by equations (42), since otherwise the rigid-field convention is trivial. Recently, the authors (**21**) have found that Eddington's form of the equations can be obtained if the rigid-field convention is replaced by the condition

$$\rho+kK^0 = 0,$$

where
$$\rho = \int X^{\alpha}j_{\alpha}\frac{\delta K^0}{\delta j}\,dV.$$

It can be shown that this condition is a restriction on the physical systems considered, but the physical interpretation of this restriction is not clear.

4.3. Eddington uses the results of the previous section to introduce the 'principle of the top particle'. In the discrete case, we have that if H^0 is a homogeneous function of degree n in the occupation factors, then $H^0 = (1/n)E^0$. Now, from (28), we have, in the scale-free case, $H^0 = -kE^0$, from which Eddington concludes that H^0 is a homogeneous function of degree $-1/k$ in the pseudo-discrete occupation factors $j_r = j(X^{\alpha})\,d\tau_r$. With this quite unjustified assumption, Eddington goes on to consider an assemblage of particles in one pseudo-discrete state, so that H^0 varies as $j^{-1/k}$ and j is proportional to the particle density s. Hence,

$$H^0 \propto s^{-1/k}, \qquad \frac{dH^0}{ds} = -\frac{1}{k}\frac{H^0}{s}. \tag{48}$$

Consider a unit volume, choosing the unit large enough for s to be very large. If one particle (per unit volume) is removed from the assemblage, the amount of H^0 removed is dH^0/ds, and so Eddington calls dH^0/ds 'the H of the top particle' and denotes it by $\mathfrak{H}$. The average H^0 per particle (in unit volume) is H^0/s, which he calls 'the H of the mean particle' and denotes it by $\bar{H}$. Then, from (48),

$$\mathfrak{H} = -\bar{H}/k.$$

Since this is similar to (28), Eddington applies this result to the energy tensor and states that for an assemblage of particles in an initial pseudo-discrete state:

> *The particle and total energy tensors are respectively the energy tensors of a top particle and mean particle.*

Apparently Eddington's meaning is that the energy of a top particle is the energy of each particle of the object system while the energy of a mean particle is the total energy of the system averaged over each particle of the system; i.e. if E^0, H^0 are the particle and total energies respectively of an object system consisting of n particles, then E^0/n and H^0/n are respectively the top particle and mean particle energies. Although there are a number of errors in Eddington's argument, the idea of the top particle seems interesting and can be explained in the following way:

The particle which is selected as our object-particle is initially an unidentified member of a large assemblage, i.e. it is an average, or mean, particle. In this state it has associated with it a certain energy. On selection as the object-particle, it is still associated with the same energy, but this energy is now partitioned between the particle and its complementary object-field. If H^0 is the energy of the mean particle (or the total energy of the object-particle) and E^0 is the particle energy of the object-particle, then the energy partition is given by (35), i.e.

$$H^0 = -\theta E^0.$$

If we select any particle from the assemblage, it is equally likely to belong to any energy level, but Eddington pictures the particles of the assemblage arranged in energy levels which are built up successively as j increases from 0 to its actual value. Each energy level must be completely filled before we proceed to the next one, so that we eventually have a series of energy levels which are fully packed except for the top level. The particles in the top level (top particles) can take part in quantum transitions, but the particles in the fully packed energy levels do not take part in these transitions and so form the rigid background field that is required. If we select a particle it must come from the top level, since we must pack each energy level before we start to fill the next. (This idea is somewhat similar to the exclusion principle in classical quantum theory.) Hence, our selected particle (object-particle) is always a top particle.

As before, we take the particles to be simple scale-free particles characterized by an energy-tensor and nothing else. If the initial state is the state of almost exact rest, the energy tensor reduces to a single component, the density, and we have the relation

$$\rho_{\text{mean}} = -\theta\rho_{\text{top}},$$

where ρ_{mean} and ρ_{top} are the proper densities of the mean and top particles respectively. Since the selection of a particle as an object-particle does not alter the proper volume associated with it when it was a mean particle, it follows that the proper masses are in the same ratio as the proper densities and we have

$$m_{\text{mean}} = -\theta m_{\text{top}}. \tag{49}$$

An alteration in the multiplicity factor k produces an alteration in the partition of the total energy tensor $T_{\mu\nu}$, but leaves $T_{\mu\nu}$ unaltered. Hence, in (49), an alteration in k would leave the mean particle unaffected, but the energy tensor and mass of the particle are changed in inverse ratio to θ. A quantum particle of multiplicity k is a V_k and since quantum particles are top particles we have that the relation between the masses m_1, m_2 of a V_{k_1} and V_{k_2} is

$$\frac{m_1}{m_2} = \frac{\theta_2}{\theta_1}. \tag{50}$$

Note that since we are dealing with the ratio of masses we are within the bounds of scale-free theory.

The large assemblage of particles in a state of almost exact rest which has been considered here is the standard uranoid of scale-free physics. A small number of unidentified particles are selected as the object-system, and these automatically become the top particles. The rest of the assemblage forms the environment, and is postulated to remain in its initial state of almost exact rest whilst the object-particles have freedom of transition to other states. Since the particles of the environment only need to be treated collectively, they can be taken to be mean particles. Hence, an alteration in k does not alter the environment since such an alteration does not affect the mean particle. Accordingly, Eddington claims that (50) applies to V_{k_1} and V_{k_2} particles in the same environment and m_1/m_2 is a genuine mass-ratio as would be determined by practical measurement.

4.4. In **FT**, § 18, Eddington considers the mass-ratio of the proton and electron. We have shown in section 4.2 that there are two types of energy to consider, namely, initial and transition energies. Eddington attempts to simplify the analysis by assuming that initial and transition energies are carried by different particles. He introduces the following types of particles:

Initial particles, for which no change of state is contemplated. They are generally taken to be in a state of almost exact rest.

Transition particles, which have many possible states. Their initial state is a state of zero energy tensor. The names and properties of these particles are obvious from the relation

$$E_{\mu\nu} = (E_{\mu\nu})_0 + \delta E_{\mu\nu},$$

where $(E_{\mu\nu})_0$ is the energy of the initial particle and $\delta E_{\mu\nu}$ is the energy of the transition particle.

This procedure is the same as that used in classical mechanics where the system is replaced by an *external particle* moving with the centre of mass and having the total mass of the system, together with *internal particles* describing orbits relatively to the centre of mass and having suitable reduced masses. The rest energies of the internal particles are zero, since the total mass is carried by the external particle. Hence, by choosing a coordinate frame in which the centre of mass is at rest, the external particle is an initial particle and the internal particles are transition particles. Eddington considers the special case of a two-particle system, the masses of the components being m and m'. This system may be transformed into an external mass M moving with the centre of gravity and an internal mass μ specifying the relative orbit, where

$$M = m+m', \qquad \mu = \frac{mm'}{m+m'}.$$

Note that the rest energy of the internal particle is zero, not μ, so that it is a carrier of transition energy only, while the external particle carries initial energy only, provided changes in the motion of the centre of gravity of the system are not contemplated. Eddington applies this to a hydrogen atom, remarking that it may be split into an external particle and an internal particle, and it may also be split into a proton and an electron. But the latter method is confusing if we want to study the properties of the atoms as a whole, although it is necessary when the results for hydrogen are made the basis for the theory of more complete systems.

By applying the division into internal and external particles to a hydrogen atom, Eddington calculates the ratio of the masses of the proton and the electron. However, his method is based on his partition equations (28) and also involves a confusion of scale-fixed and scale-free characteristics, so his result is questionable. We will now present an improved form of his argument, and show that it leads to loss of the numerical result.

Since the probability distribution of the proton and electron in a hydrogen atom extends over the same volume, we may assume that the internal and external particles also occupy the same volume. Then instead of expressing the division into internal and external particles by masses, we may express it in terms of densities. If ρ_M, ρ_μ are the densities of the internal and external particle, and ρ, ρ' the densities of the components of the system (i.e. proton and electron), then we have

$$\rho_M = \rho+\rho', \qquad \rho_\mu = \frac{\rho\rho'}{\rho+\rho'}.$$

In **FT**, § 17 (see Appendix I), Eddington shows that a particle (or carrier) that is described by an energy tensor and nothing more has 136 degrees of freedom, i.e. it is a V_{136}. Such a particle is known as a standard carrier and is composite, since it contains an internal stress system. When this internal stress is zero, the particle reduces to a simple particle which is a V_{10}.

The hydrogen atom is a composite particle, so we take it as a V_{136}. By dividing it into an external and internal particle, we replace the V_{136} by two simple V_{10} particles, one of which carries internal energy only, while the other carries transition energy only. In terms of wave analysis, the double wave function Ψ of the composite particle is factorized into wave functions ψ_e, ψ_i of two simple particles. Not every state of the V_{136} is a combined state of two V_{10}'s, but we are here considering a transition to a separable state. Stabilization is introduced when the 136-dimensional element $d\tau$ is replaced by the product $d\tau_e\, d\tau_i$ of two 10-dimensional elements defining pseudo-discrete states of the V_{10} particles.

Consider a distribution of s standard particles per unit volume at almost exact rest. If ρ_0 is the total proper density, then the density apportioned to the particles is

$$\rho_{136} = E_{44} = -\frac{T_{44}}{\theta_{136}} = -\frac{\rho_0}{\theta_{136}}.$$

Consider now a distribution of s external particles and s internal particles per unit volume at almost exact rest. The internal particles make no contribution to the energy, since they are at rest. The total energy remains unchanged when k changes, so it is still ρ_0. The density apportioned to the external particles is thus:

$$\rho_{10} = -\frac{\rho_0}{\theta_{10}}.$$

If we introduce a small transition, the transition energy is attributed entirely to the internal particles. This transition energy does not change as θ changes; it remains as $\delta T_{\mu\nu}$, so that it is always associated with ρ_0. Hence, we have that the hydrogen atoms can be considered as two systems of V_{10} particles, external and internal particles. The external particles are carriers of initial energy and are accordingly attributed a density of $-\rho_0/\theta_{10}$, while the internal particles are carriers of transition energy only and are accordingly attributed a density ρ_0.

In our notation we have

$$\rho_M = -\frac{\rho_0}{\theta_{10}}, \qquad \rho_\mu = \rho_0,$$

and also by writing ρ, ρ' as ρ_p, ρ_e, i.e. densities of proton and electron, we have

$$\rho_M = \rho_p+\rho_e, \qquad \rho_M \rho_\mu = \rho_p \rho_e,$$

so that ρ_p, ρ_e are the roots of the equation

$$\rho^2-\rho\rho_M+\rho_M\rho_\mu = 0.$$

This compares with Eddington's equation

$$m^2-mM+M\mu = 0.$$

Substituting for ρ_M, ρ_μ this becomes

$$\rho^2+\frac{\rho\rho_0}{\theta_{10}}-\frac{\rho_0^2}{\theta_{10}} = 0,$$

and the roots ρ_p, ρ_e of this equation are in the ratio

$$\frac{\rho_p}{\rho_e} = -\left\{\frac{L_{10}}{20}+1+\frac{L_{10}}{20}\sqrt{\left(1+\frac{40}{L_{10}}\right)}\right\}.$$

Since the probability distribution of the proton and the electron extend over the same volume, it follows that their masses are in the same ratio as their densities,

$$\frac{m_p}{m_e} = -\left\{\frac{L_{10}}{20}+1+\frac{L_{10}}{20}\sqrt{\left(1+\frac{40}{L_{10}}\right)}\right\}.$$

It is found that this ratio will have the same value as Eddington finds:

i.e. $$\frac{m_p}{m_e} = 1847{\cdot}60,$$

if $L_{10} = -(136)^2$. But, of course, there is no guarantee that this will be so. The value of L depends on k and on the functional form of K^0, and since we do not know the form of K^0, this value of L cannot be calculated. We suggest that the value of L (or θ) may be found by an experimental procedure at some stage, which means that Eddington's theory is, in fact, empirical and not epistemological as he claimed; and, of course, none the worse for that.

4.5. Eddington's partition equations (28) suggest that a V_{136} is associated with a field energy which is -137 times its own energy, so that the particle is supposed to be in a gravitational field far outside our ordinary experience. However, gravitational fields may be created by a transformation of coordinates and Eddington asserts that the apparently intense fields postulated in scale-free physics must be furnished in this way. The rigid-field convention is restated in the form: the coordinates must be so chosen that the field is stationary for small changes of the occupation factors. Such coordinates are called 'rigid coordinates'. Eddington then determines the rigid coordinates for a uniform distribution of particles in an initial state of almost exact rest, and, by means of a somewhat doubtful argument, finds that the rigid coordinates (x,y,z,t) are related to the Galilean coordinates (x',y',z',t') by

$$x' = x, \quad y' = y, \quad z' = z, \quad t' = -kt.$$

He concludes from this that in wave mechanics we must employ a time t which is $1/k$ times the Galilean time t', but this result has only been proved for particles which are at rest and a more general result is not necessarily true.

Our version of the partition equations (42) makes the idea of rigid coordinates both unnecessary and unlikely. For a V_{136} we have

$$W_{\mu\nu} = -(\theta_{136}+1)E_{\mu\nu},$$

and the value of θ_{136} may be such that $(\theta_{136}+1)$ is of reasonable proportions.

By following Eddington's argument, we find that the rigid coordinates (x,y,z,t) for a distribution of particles at rest are given by

$$x = x', \quad y = y', \quad z = z', \quad t = -t'/\theta.$$

Since the particles are at rest, there are no transitions and θ is effectively constant. If, like Eddington, we generalize this result, we would conclude that in wave mechanics a time t which is $1/\theta$ times the Galilean time t' must be employed.

This result means that the time coordinate changes with every change in the occupation factors of the eigenstates, i.e. in order that the field should be rigid we must employ a different set of coordinates after each transition. This idea seems absurd and since the idea of rigid coordinates is unnecessary with our partition equations, we will not consider using these coordinates.

4.6. When we require the measurement of some characteristic of an object-body, we do not make independent observations of the object-body and a reference body. Instead the object-body and the reference body are observed as a joint body, and any measurement made determines characteristics which belong to both jointly. However, these characteristics are usually allotted entirely to the object-body, or partitioned between the two bodies in some self-consistent manner. In **FT**, § 22, Eddington remarks that the language of physics is bound up with the convention of substituting self properties for mutual properties and that we must accept it in principle. This reference to 'bodies' is picturesque, but somewhat confusing. The fact is that a complex abstract structure is needed to describe both a system and its environment interacting in an experiment. Eddington contends that it is usual (probably because it is easier to dodge the complexity) to describe this complex structure in terms of the simple ones, by allotting the numerical characteristics to them.

Eddington considers the object-particle and reference particle ('comparison particle') to be simple so that their mechanical characteristics are fully specified by complete momentum vectors p_μ, p'_μ. Here again, Eddington appears to be making the error of describing scale-free particles in terms of scale-fixed characteristics (cf. section 3.2); instead of using p_μ, he should use the root vector $\sqrt{\rho_0}\, v_\mu$ (**FT**, § 17, *see* Appendix I). He now asserts that the mutual energy tensor is necessarily of the form

$$M_{\mu\nu} = \tfrac{1}{2}C(p_\mu p'_\nu + p'_\mu p_\nu),$$

where C is a pure number, for there is no other tensor of the second rank, having the dimensions of an energy tensor, depending symmetrically and inseparably on the mechanical characteristics of the two particles. This is an error since, for example, an energy tensor of the form

$$M_{\mu\nu} = \tfrac{1}{2}C\frac{p_\alpha p'^\alpha}{\sqrt{\{(p_\beta p^\beta)(p'_\gamma p'^\gamma)\}}}(p_\mu p'_\nu + p'_\mu p_\nu)$$

also satisfies this requirement. The required energy tensor necessarily has the factor $(p_\mu p'_\nu + p'_\mu p_\nu)$, but it may be multiplied by any dimensionless symmetrical combination of the p_α and p'_α. However, Eddington is concerned only with rest energies in his argument, and in this case the dimensionless factor becomes 1 (or a numerical constant).

Similarly, Eddington asserts that the self-energy tensors must be of the form

$$T_{\mu\nu} = Ap_\mu p_\nu, \qquad T'_{\mu\nu} = A'p'_\mu p'_\nu,$$

and from this deduces that, for particles at almost exact rest,

$$Am^2+A'm'^2 = Cmm'. \tag{51}$$

Now in **FT**, § 18, Eddington finds that the masses of the proton and electron, m_p, m_e, are the roots of the equation

$$10m^2-136mm_0+m_0^2 = 0, \tag{52}$$

where m_0 is the mass of a standard carrier. By comparing these last two equations, he deduces that

$$m' = m_0, \qquad A:A':C = 10:1:136.$$

This leads him to the conclusion that the object, comparison and mutual bodies are of multiplicities 10, 1, and 136 respectively. This is, of course, the expected result since the mutual body is a composite particle, i.e. a V_{136}; the object body is a simple particle, i.e. a V_{10}; and the comparison body is a carrier of the standard of reference, so that it has only one degree of freedom and so is a V_1. However, Eddington's argument is wrong, not only for the reasons already given, but also because the comparison between (51) and (52) cannot be made, since the latter was deduced by considering a composite particle to be divided into two simple particles, one of which is a carrier of initial (i.e. rest) energy only, while the other carries transition energy only. In the present case, the composite particle is divided into two particles both of which carry rest energy, so there is no comparison between the two cases.

On attempting to correct this argument, we find, as usual, that the numerical result disappears. If ρ_0, ρ'_0, ρ_{0_m} are the total proper-densities associated with the object-particle, comparison particle and mutual body, respectively, and θ, θ', θ_m the θ's associated with these three bodies, then by taking the mutual energy tensor to be of the form

$$M_{\mu\nu} = -\tfrac{1}{2}\theta_m(\surd\rho\, v_\mu \surd\rho'\, v'_\nu+\surd\rho'\, v'_\mu \surd\rho\, v_\nu)$$

we find that for a state of almost exact rest

$$\rho_{0_m} = -\theta_m\surd(\rho\rho').$$

Similarly, the self energies of the particles give rise to

$$\rho_0 = -\theta\rho,$$
$$\rho'_0 = -\theta'\rho'.$$

Note that the expression $\frac{1}{2}(\surd\rho\, v_\mu \surd\rho'\, v'_\nu + \surd\rho'\, v'_\mu \surd\rho\, v_\nu)$ is the particle energy of the mutual body.

These expressions lead to the ratio of the densities of the object-particle and comparison particle in the form

$$\frac{\rho}{\rho'} = \frac{\theta_m^2}{2\theta^2} - \frac{\theta'}{\theta} \pm \frac{\theta_m^2}{2\theta^2}\sqrt{\left(1 - \frac{4\theta\theta'}{\theta_m^2}\right)},$$

but no numerical result follows since the values of the θ's are unknown.

Eddington also introduces the idea of a comparison hole, which is a source of negative energy such that when a system consisting of an object-particle and its comparison hole is superimposed on the uranoid the density is unchanged. In this case there is no mutual body.

4.7. This argument 'B' of **FT** has many interesting aspects. It suffers from using Eddington's equation $T_{\mu\nu} = p_\mu p_\nu$ which results in the confusion of scale-free and scale-fixed quantities, and also from his form of energy partition equations, which are made trivial by the rigid-field convention. On correcting these equations in the obvious way, Eddington's numerical results are lost, which indicates that his theory does, in fact, require empirical data. But the backbone of argument 'B' is an attempt to discuss quantitatively the relation between physical quantities in two abstract structures, and this is what gives the argument its interest. For this reason, and because we shall have further use for the relation later, it seems worth while to consider the equation for $T_{\mu\nu}$ again.

The crucial step in Eddington's opinion is the recognition of *scale*. If we are to have a theory about protons and electrons it must recognize their difference from dust-particles and cricket balls. This difference is expressed in orthodox quantum mechanics in terms of the quantization of their energies and angular momenta. Eddington takes seriously the fact that the atomic particles are small. Just as $\hbar$ enters for the energies, so here a scale constant σ must be introduced. (Alternatively one could introduce a mass m, and define $\sigma = \hbar/m$; this would be more usual (with m as the electron mass). But Eddington leaves himself some freedom by introducing σ and not relating it immediately to $\hbar$.) Ordinary macroscopic descriptions are scale-free, and the simplest experimental conditions there are described in terms of an energy tensor $T_{\mu\nu}$. Such an energy tensor has normally 10 degrees of freedom; but for a cloud of dust it can be analysed into a root vector, since

$$T_{\mu\nu} = \rho v_\mu v_\nu = (\surd\rho\, v_\mu)(\surd\rho\, v_\nu).$$

This analysis does not affect the scale-free nature in the least. But the degrees of freedom become 4.

Can such a scale-free system be analysed into elementary particles, i.e. into scale-fixed systems? Such an analysis will be of a different kind. Eddington's contention is that this energy tensor *will* correspond to an assembly of elementary particles of momenta p_μ; and the equation

$$T_{\mu\nu} = Ap_\mu p_\nu,$$

where $A = D(c^4/8\pi\gamma\hbar^2)$, is a *rule* for finding these particles. Do the difficulties of section 3.2 still apply to this interpretation of the equation? The first difficulty of the domain of definition is avoided if we have an assembly of particles, but then we should strictly write

$$T_{\mu\nu} = A \int n(p_\mu)p_\mu p_\nu \, d^4p,$$

where $n(p_\mu)\, d^4p$ is the *number* of particles per unit volume with a given momentum p_μ. Such an energy tensor is not then a direct product; and the difficulty noted in section 4.6 that p_μ is scale-fixed and $T_{\mu\nu}$ scale-free is also no longer a difficulty.

The constant A must be inserted to allow the freedom of analysis into particles which Eddington uses; for we could split an assembly of rather complex particles into one of double the number each carrying half the variables, so that

$$A'n' = An, \qquad A' = \tfrac{1}{2}A.$$

In other words, apart from any dimensional constants and so on, A must include as a factor the number of degrees of freedom. It is by no means the case that the energy tensor will be that calculated on the assumption that the elementary particles formed a cloud of dust with the same number of particles; this would require, in the case when the particles were almost at rest, that $Am_0 = 1$, where $m_0^2 = p^\alpha p_\alpha$. This result (since A is proportional to the number of degrees of freedom k) is consistent with Eddington's fundamental formula

$$\frac{m_1}{m_2} = \frac{k_2}{k_1},$$

which we were unable to deduce by his methods in section 4.3. The real significance of this equation is now seen to be that, if the complexity of the particles is halved, so that n is doubled, A must be halved and *if* we are to preserve the 'dust-particle analysis' then m_0 must be doubled. Naturally, it is not always convenient to preserve the 'dust-particle analysis', so that the formula $Am_0 = 1$ is not always true.

It remains to consider the status of Eddington's equation

$$T_{\mu\nu} = Ap_{\mu}p_{\nu}.$$

It can be considered an arbitrary convention which serves merely to define our scale-fixed physical entities. But this is to ignore the uses which Eddington makes of it, by assuming that these scale-fixed entities are indeed the particular ones (protons and electrons) which he considers. The rule really has the status of a scientific hypothesis which is open to falsification by experiment; for from the joint assumptions of the rule and the fact that the scale-fixed entities are protons and electrons, we can deduce various results which can be tested. Considered as such a hypothesis it does not suffer from the criticisms which we have made in the preceding sections.

5

THE NON-COULOMBIAN POTENTIAL

5.1. We have now described the two basic arguments 'A' and 'B' of Eddington's theory. We proceed to explain the secondary arguments which apply the basic conclusions to physical problems. The simplest secondary argument in **FT**, which in fact rests only on the basic argument 'A', is the calculation of the energy of a pair of protons. We have already said something about this argument in section 2.6, but it is convenient to repeat it here with the fuller discussion. Eddington sees a two-particle system in which the particles have opposite charges as an elementary one, because it can be put into a neutral environment. If the charges have the same sign the environment will be polarized by them, and so there will be opposite charges somewhere in it. Here we see how the investigation of the effect of the environment divides two-particle systems into two classes in a very precise manner. Now, the Coulomb energy of the proton-electron system, according to Eddington, has its origin in some kind of interchange, and even if we do not accept his arguments as a whole the value $e^2/\hbar$ of the fine-structure constant certainly seems to connect the Coulomb energy with some quantum-mechanical phenomenon. However, the proton-proton system would apparently give rise to quite a different Coulomb energy because there are no steady states. Accordingly, Eddington calculates the Coulomb energy for this system by an independent argument based on the assumption that a charged particle has no Coulomb energy if it is in a neutral environment. 'Neutral' here implies equal distributions of protons and electrons. By an elementary argument it follows that the proton-proton energy must be equal and opposite to the proton-electron energy. He then goes on to question this argument in the case when the two protons are very close together so that the measured distance between them is zero. His basis for this is that the two like charges certainly cannot be superposed for the reasons which we have stated, and so some adjustment of the energy must be made if we do not put in the induced charges. On the other hand, when the separation vanishes no such change can be made. It is not easy to see why we are allowed to make such a change, even when the separation does not vanish, but this is the assumption made by Eddington, and he makes the further assumption that the form of the contribution to the energy for the vanishing of the separation is that of a delta function of the measured separation. Of course, it would be possible to assume other improper functions as contributions. The delta function is a rather arbitrary choice. Once the delta function has been chosen we can convert the delta function of the measured distance into a certain Gaussian distribution of the calculated distance as in the argument 'A', and this gives rise to a non-Coulombian energy.

5.2. We find this argument of Eddington's very unconvincing, and we suggest replacing it by the following argument, based on some of his ideas, which relates the non-Coulombian energy directly to the ordinary Coulomb energy. More precisely, we shall consider the effect of the uncertainty of the origin on the problems of the connexion between action-at-a-distance theories and field theories. We shall take the view that Eddington has drawn an important distinction (**FT**, § 5) between measured and calculated distances, but that he has failed to apply this distinction correctly. If we correct his application of this distinction, for the particular case of a scattering problem between particles under a Coulomb field, the result of this calculation is that to a certain degree of approximation the Coulomb field is amended by a potential of Yukawa type, $Ae^{-r/r_0}/r$. The range-constant r_0 is determined in terms of σ, but the strength A has to be determined by other arguments.

It should be emphasized that the Yukawa potential arises entirely from the Coulomb field when we take account of the uncertainty of the origin. None the less it is natural to seek to employ it to account for nuclear scattering. Two difficulties are met with, however; a mathematical difficulty is that the Yukawa potential is derived by using an approximation which is only valid for certain particles. We overcome this difficulty by using requirements of relativistic invariance to generalize the result we have already obtained to the case of nucleons. The range constant is then of the right order to account for low-energy proton-proton scattering, and can be adjusted to the correct strength by choosing an arbitrary constant, of the dimensions of mass, to be of the order of 70 electron masses.

5.3. The other difficulty is much more interesting. Eddington's calculations (which we believe to be erroneous) lead him to conclude, from the distinction between measured and calculated distance, that the nuclear potential should have the Gaussian form Ae^{-r^2/r_0^2}. He therefore concludes (**FT**, p. 10) that 'we need not hesitate to reject the meson-field hypothesis altogether'. In the light of our calculation, and the present position in meson-field theory, we cannot be so dogmatic.

We may remark that, since our Yukawa potential is derived from the electrostatic Coulomb potential, it predicts no force between neutrons and protons. But experiment shows the nucleon interaction to be charge-independent. Moreover, at higher energies it appears to be important to consider tensor coupling, and to take account of spin terms in the potential. One is at first tempted to suppose that what has been shown in the present working is that the objective of meson theory is not to explain the discrepancy between Coulomb and actual scattering, but only that between (Coulomb+Yukawa) and actual scattering. This in itself would be an advance, since meson theory raises very difficult mathematical problems, and these would then have to be solved only to a degree of approximation needed to account for a smaller discrepancy. But it would be very surprising if one had to use the present theory for proton-proton scattering, and meson theory as it stands for neutron-proton scattering.

On the other hand, it is clear that the theory as it stands cannot account for high energy scattering. From the point of view of Chapter 1, we would say that it is too restricted an abstract structure for this purpose. A more complex abstract structure must be created, but the evidence under discussion suggests that this will have little in common with meson field theory. Thus we appear to have here a state of affairs in which both meson-field theory (if it proves satisfactory) and the elaboration of the present theory (if it can be constructed) would give substantial numerical agreement about nuclear properties, although they make quite different assumptions. We have described in Chapter 1 how we can have chains of abstract structures, simply ordered with respect to complexity. The instance of the present theory and a meson-field-theory explanation of nuclear properties is one of two theories in different chains and so not ordered with respect to complexity. It may be that the disjoint nature of Eddington's work and meson-field theory is at the root of his attitude, which gives meson theory very short shrift.

5.4. In classical electrodynamics, the following process is familiar:

The Maxwell equations in free space have for the static field of a charge e at rest at the point $\mathbf{r} = \mathbf{r}_1$, the solution

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi, \qquad \phi = \frac{e}{|\mathbf{r}-\mathbf{r}_1|}.$$

This equation is then the corresponding 'action-at-a-distance' formulation of the field equations.

Now Eddington suggests in 'A' that even the most primitive form of quantum theory requires a 'physical origin' from which to measure observable coordinates. This physical origin is distributed about the 'geometrical origin' with standard deviation σ. If the Maxwell equations are to contain observable quantities **E**, **H**, **J**, the coordinate-system used must be the physical one. The coordinates are measured coordinates, and the analytical theorem, which is still true, that $1/r$ provides the static solution, refers to the *calculated* value of $r = |\mathbf{r}-\mathbf{r}_1|$.

On the other hand, the usual method of calculating Coulomb scattering problems is as follows (**18**):

In the Schrödinger equation the potential ϕ is given the value e^2/r. Usually the origin is chosen at the centre of gravity, but by an ordinary transformation (**18**, pp. 101–2) it can be moved to the scattering particle. A solution is now sought in terms of r and angular coordinates.

The origin having been chosen on the scattering particle, the coordinate which enters here is the measured radius vector r' (say), but the potential is still e^2/r. The problem can only be solved, therefore, when V is found in terms of r'. We observe that we have here a very intractable double probability problem, since the relation between r and r' is a statistical one, and the Schrödinger equation is in any case a statistical equation. An obvious approximation to seek is to replace V by its expectation value in terms of r', and then to solve the resultant ordinary Schrödinger equation. We can see physically that there will exist a value of the mass, m_0 say, of the scattering particle for which this is a good approximation. For as the mass varies from a sufficiently large to a sufficiently small value, the uncertainty with which the position of the scattering particle can be measured varies in the opposite way, and so passes through the value σ. We shall call particles of mass m_0 'standard particles', without meaning to imply that they are to be found in nature.

5.5. For standard particles, then, if we consider any given measured radius r', r has standard deviation $\sigma\sqrt{2}$, and so has frequency function

$$f(r) = Ae^{-(r-r')^2/4\sigma^2}.$$

Hence the expectation value of the Coulomb potential per unit charge is

$$\bar{\phi} = \frac{\int_0^\infty (1/r)\,.\,4\pi r^2 e^{-(r-r')^2/4\sigma^2}\,dr}{\int_0^\infty 4\pi r^2 e^{-(r-r')^2/4\sigma^2}\,dr}.$$

Putting $r' = 2\sigma y$, we easily have for the value of $2\sigma\bar{\phi}$:

TABLE 1

y	$2\sigma\bar{\phi}$	y	$2\sigma\bar{\phi}$
0	1·128	0·9	0·724
0·1	1·075	1·0	0·690
0·2	1·023	1·1	0·658
0·3	0·974	1·2	0·627
0·4	0·927	1·3	0·599
0·5	0·882	1·4	0·572
0·6	0·839	1·5	0·547
0·7	0·799	..	..
0·8	0·760	2·0	0·445
..	..	2·5	0·370

5.6. In calculation of scattering of standard particles, then, the expectation value $\bar{\phi}$ is to be used instead of the Coulomb potential, $1/2\sigma y$. If we do use the Coulomb potential, the results will be in error unless we subtract from it the potential ϕ_1, where

$$\phi_1 = \frac{1}{2\sigma y} - \bar{\phi} = \frac{1-2\sigma y\bar{\phi}}{2\sigma y}.$$

The potential ϕ_1 is then analogous to the non-Coulombian potential. From Table 1, however, it is easy to verify the following very striking approximation to ϕ_1:

$$2\sigma\phi_1 \doteqdot \frac{e^{-1\cdot 17y}}{y}.$$

We have in fact

TABLE 2

y	$2\sigma\phi_1$	$(1/y)e^{-1\cdot 17y}$	y	$2\sigma\phi_1$	$(1/y)e^{-1\cdot 17y}$
0·01	98·88	98·84	0·9	0·387	0·388
0·1	8·92	8·90	1·0	0·310	0·310
0·2	3·98	3·96	1·1	0·251	0·251
0·3	2·36	2·35	1·2	0·206	0·205
0·4	1·57	1·57	1·3	0·170	0·168
0·5	1·12	1·11	1·4	0·142	0·139
0·6	0·827	0·826	1·5	0·119	0·115
0·7	0·630	0·630	..	..	..
0·8	0·490	0·490	2·0	0·054	0·048
..	..	..	2·5	0·030	0·021

We may notice the very good agreement in the range from $y = 0{\cdot}01$ to $y = 1{\cdot}5$. The analytical basis of this approximation, if there is one, is unknown. The range of agreement is the important one in scattering, so long as the energy is not so high as to cause much penetration to $y < 0{\cdot}01$, since for $y > 1{\cdot}5$ the potential ϕ_1 is small compared with the Coulomb potential.

We now wish to compare the potential ϕ_1, with that which accounts fairly well for the scattering of protons by protons. This section is concerned only with orders of magnitude. The potential ϕ_1 is of the form $(1/r')e^{-r'/r_0}$, where $r_0 = 1{\cdot}70\sigma$. If we adopt, for the moment, the value of σ obtained from the 'planoid', i.e. the standard abstract structure in which the environment is a sphere of radius R,

$$\sigma = \frac{\mathrm{R}}{\sqrt{(5N)}},$$

we have $V_0 r_0 = 0{\cdot}1/\sqrt{N}$, giving a relation between r_0, and the limiting Hubble constant. In more convenient units, if V_0 is expressed in km/sec/megaparsec, and r_0 in 10^{-13} cm, then with Eddington's value of N we have

$$5V_0 r_0 \doteqdot 10^3.$$

Since V_0 is somewhat less than 100, r_0 will be about 2; the experimental results on protons give about 2·1. It is, however, most surprising that even the order of magnitude should be correct. For the strength, however, we get $A = e^2/r_0$, where e is the electron charge, and this gives about 0·6 MeV, a result in error by about 40:1. Actually A, r_0 are not known so well separately, but Ar_0^2 is known closely. Our calculated value is $2{\cdot}6\times 10^{-26}$ against the observed value of 62×10^{-26}. These calculations are not very accurate, and a more complete comparison of the theory with experiment is needed. But even at this stage it appears fairly clear that the standard particles involved cannot be protons.

5.7. We shall now attempt to complete the theory by finding the non-Coulombian potential for non-standard particles. We do not wish to try to solve the problem of the double probability distribution, so we suggest as an alternative course that we investigate in what ways the theory can be generalized so as to be Lorentz invariant. We first consider the matter classically.

Various forms of relativistic particle dynamics are possible. A choice between them can be made, however, by using the algebraic theory of the second part of **FT**. As we have not developed this we shall simply refer in the following discussion to the results that the algebraic theory provides.

Let us assume firstly that the equation of motion of a particle has the form

$$\frac{dv^\alpha}{ds} = G^\alpha.$$

Since $v^\alpha v_\alpha = 1$, we have $G^\alpha v_\alpha = 0$, so that G^α must depend on v^α. The two forms

$$G^\alpha = H^{\alpha\rho} v_\rho, \qquad G^\alpha = (\delta^\alpha_\rho - v^\alpha v_\rho) H^\rho$$

(where $H^{\alpha\rho} + H^{\rho\alpha} = 0$) are completely equivalent (e.g. one may write

$$H_\mu = H_{\mu\nu} v^\nu, \qquad H_{\mu\nu} = H_\mu v_\nu - H_\nu v_\mu),$$

and quite general if $H^{\mu\nu}$, H^μ are allowed to depend on v^α. The simplest cases to consider are

(i) fields of type I, where $G^\alpha = H^{\alpha\rho} v_\rho$, and $H^{\alpha\rho}$ is independent of v^α;

(ii) fields of type II, where $G^\alpha = (\delta^\alpha_\rho - v^\alpha v_\rho) H^\rho$ and H^ρ is independent of v^α.

A field which falls into neither of these classes may be called of type III, but we shall not consider these further.

The quantities $H^{\alpha\rho}$, H^ρ are not field strengths, since they depend on the particle (even if not on its velocity). We can however determine how they depend on the field quantities by supposing that field is determined, by using an appropriate algebra, in terms of flux from its sources. It then turns out that for fields of type I, acting on spherically symmetric particles, we must have

$$H_{\mu\nu} = aF_{\mu\nu} + bF^*_{\mu\nu},$$

where a, b might depend on the invariants of $F_{\mu\nu}$, but do not do so in the case when the effect of the field is additive. The field $F_{\mu\nu}$ then satisfies equations like Maxwell's.

For fields of type II, on the other hand, a corresponding argument gives the result

$$H_\mu = aF_\mu = -a\,\partial_\mu \psi,$$

where a may depend on the magnitude of F_μ, but in the simplest case does not.

Let us begin by considering how such equations can describe the gravitational field. In such a case the principle of equivalence makes a, b constants, and it is simplest to take $a = 1$ (and $b = 0$ for type I). The type I equation of motion is then

$$\frac{dv^\mu}{ds} = F^{\mu\rho} v_\rho,$$

and if we write this as

$$\frac{d}{ds}(mv^\mu) = mF^{\mu\rho} v_\rho,$$

m is just a constant parameter (the rest-mass). But for type II we have

$$\frac{dv^\mu}{ds} = F^\mu - v^\mu v_\nu F^\nu = -\partial^\mu \psi + v^\mu \frac{d\psi}{ds},$$

so that it is natural to rewrite this in the form

$$\frac{d}{ds}(mv^{\mu}) = mF^{\mu},$$

where $m = m_0 e^{-\psi}$. In this case the rest-mass depends on the field. For slow motions and 'weak fields' (i.e. small ψ), we have for the current mass an expression

$$\frac{m}{\sqrt{(1-v^2)}} = m_0+\tfrac{1}{2}m_0 v^2-m_0\psi+...,$$

showing the effect of ψ as the classical potential energy.

For the electromagnetic and corresponding fields we expect the simplest case to arise when a is a charge-mass ratio. One therefore rewrites the equations as

$$m\frac{dv^{\mu}}{ds} = eF^{\mu\nu}v_{\nu} \qquad \text{(type I)},$$

$$m\frac{dv^{\mu}}{ds} = e(F^{\mu}-v^{\mu}v_{\nu}F^{\nu}) \qquad \text{(type II)},$$

where e is now treated as constant. The first equation is consistent with a constant value of m but the second is not, when we use the forms

$$\frac{d}{ds}(mv^{\mu}) = eF^{\mu\nu}v_{\nu},$$

$$\frac{d}{ds}(mv^{\mu}) = -e\partial^{\mu}\psi.$$

In the second equation we have $m = m_0-e\psi$. It is the electromagnetic field in which we are interested here, but we included the gravitational discussion for later use (Chapter 7).

5.8. It is clear that the only convenient ways in which G^{α} can represent the non-Coulombian field (derived by differentiating the non-Coulombian potential ϕ_1) are therefore:

(i) ϕ_1 is taken as the fourth component of a 4-vector, in the same way as the Coulomb potential, and the rest-mass is constant;

(ii) ϕ_1 is chosen as $e\psi$ in type II above, and the rest-mass is of the form $m_0-e\psi$.

Let us now consider particles of spin $\frac{1}{2}$. The Dirac equation for the free particle is

$$\gamma^{\mu}\partial_{\mu}\psi = \frac{im}{\hbar}\psi.$$

The connexion of this equation with the Schrödinger equation is then well known: the Klein–Gordon equation which results on iteration gives the Schrödinger equation when a certain approximate solution is found. It is also well established that the method of taking account of a field like that of case (i) is to add the 4-potential (apart from a constant) to the differential operators. The case of a field of type II is not well known; it is natural to guess that it should be taken account of by modifying the remaining term, so as to derive

$$\gamma^{\mu}\partial_{\mu}\psi = \frac{i}{\hbar}(m_0-\phi_1)\psi.$$

From the form of the equations it is then clear that if the first type of nuclear field is assumed, the non-Coulombian potential will be independent of the mass m_0. On the other hand, the second kind of field requires a non-Coulombian potential proportional to the mass m_0.

Now protons and electrons both have spin $\frac{1}{2}$, but experiment shows that the non-Coulombian forces between electrons are negligible compared with the Coulombian; it follows that the first formulation cannot be correct, and accordingly we adopt the second one.

We conclude that the non-Coulombian potential is a scalar field, and that for proton-proton scattering we should use

$$\phi_1 = \frac{m_p}{m_0}\frac{e^{-r/r_0}}{r/r_0}.$$

The approximate figures then give an estimate of m_0 of the order of $\frac{1836}{40} = 45$ electron masses. We may repeat here that this determination is very rough.

Let us consider again for a moment the problem that the potential we have found is for charged particles only, although experiment shows a high degree of charge independence. The usual treatment in terms of isotopic spin is quite different from any that could be incorporated in the present theory. It would be necessary, in the present theory, to analyse the neutron, treated as a complex of proton and electron in a novel way (cf. **FT**, §§ 99, 100), and show that the proton constituents provide the non-Coulombian potential. This portion of the theory would be much more complicated, and consideration of it must be deferred.

5.9. The formulae determining the nuclear range-constant are now as follows:

$$\sigma = \frac{\mathrm{R}}{\sqrt{(5N)}}, \qquad V_0 = \frac{1}{\mathrm{R}_0}, \qquad M_0 = \frac{\pi \mathrm{R}_0}{2\gamma}, \qquad N = \frac{M_0}{m_H}.$$

The first two of these are derived in Chapter 2, the third one determines the mass M_0 of the initial Einstein universe, and the last one is based on the assumption that the assembly defining the uncertainty of the origin is in fact that of all the matter in the universe, analysed into hydrogen atoms. We write $\alpha = \mathrm{R}/\mathrm{R}_0$ and so we deduce by making c explicit, which was formerly taken as unity,

$$V_0 \sigma^2 = \frac{2\alpha^2 \gamma m_H}{5\pi c}.$$

Taking $V_0 = 5{\cdot}3 \times 10^{-17}\ \mathrm{sec}^{-1}$ (corresponding to a *limiting* Hubble constant of 180 km/sec/megaparsec), we get

$$\sigma^2 = \frac{\alpha^2}{5} \times 4{\cdot}4 \times 10^{-26}\ \mathrm{cm}^2.$$

Since $r_0 = 1{\cdot}7\sigma$, this gives

$$\frac{\sqrt{5}}{\alpha} r_0 = 3{\cdot}6 \times 10^{-13}\ \mathrm{cm}.$$

If we suppose as in Chapter 2 that $\alpha = 1$ we get

$$r_0 = 1{\cdot}3 \times 10^{-13}.$$

The surprising fact is that the result is of the right order of magnitude, and provides a specific relation between large-scale and small-scale constants. The comparison with scattering experiments is a little difficult, since the Yukawa potential is the least sensitive to variation of parameters. If we take the Yukawa potential corresponding to the mass of the π-meson, which is known to be the best value, we have (**18**, p. 153) figures of $1{\cdot}5 \times 10^{-13}$ cm for r_0, with a strength of 26 MeV.

The strength of our potential is $(m_p/m_0)(e^2/r_0)$, where e is the electronic charge, and using the measured value of r_0 (to avoid reproducing once more the error arising from our approximations) we get $0{\cdot}9(m_p/m_0)$ MeV, giving $m_p/m_0 \doteqdot 29$, that is, m_0 is of the order of 65 electron masses. This last result is very interesting, since the existence of a natural unit of mass equal to 68·5 electron masses has been suggested on empirical grounds (**22**, **23**) to account for the meson masses. Our result suggests strongly that the standard particle is the unit involved here. We can look on our approximation as showing that this standard mass has a certain property of 'transferability' in the equation, which is not shared by any other particles. This property is, in general terms, the property that a standard unit of mass should have.

5.10. The principal objection to the foregoing argument is that it applies only to charged particles. Thus the non-Coulombian energy between proton and neutron, which is known to a high degree of approximation to be the same as between two protons, is not predicted. Incidentally, the same may be said of Eddington's argument, for it is entirely based on the polarization of the uranoid by the proton charges. It is very possible, however, that these objections are not so serious as they seem, because in his algebraic theory Eddington builds up a neutron from two particles in a somewhat similar way to that in which a hydrogen atom is built up out of an internal and external particle (**FT**, § 94), so that from the point of view of his theory the basic particles are protons and electrons, and if we can explain the non-Coulombian energy for these this is as much as can be expected from such a simple abstract system.

6

THE COULOMB POTENTIAL

6.1. The investigation of the Coulomb potential really begins in § 23 of **FT**. This paragraph was the beginning of Chapter 3 of the draft of August 1943, and it clearly belongs there. Having exhausted the possibilities of dividing the energy between field and particle, Eddington is now going on to other ways of taking account of the environment; that is, of studying the relation between a complex abstract structure and a simpler one contained in it. The first of these ways is described picturesquely by reference to an experimental determination of the mass of an atom which will involve somewhere the use of the standard kilogramme. The theorist who forgets to put the standard kilogramme into the equations is castigated for his absent-mindedness in the same way as if he had forgotten to put in the atom. Naturally, the whole complexity of the standard kilogramme is not needed; we have merely to insert something which will perform its duty of providing a standard of mass. In § 23 this is done by inserting a comparison particle which can interact with the object particles. When we come to investigate the theory of the measurement of the mass, however, it is more convenient to define a new particle which includes both the object and the comparison particle. Eddington calls this new system a perfect object system. It is self-contained in the sense that the measured characteristics attributed to it involve nothing outside. The comparison particle is simplified so much that in the perfect object-system it is described by one variable which is now called the scale (cf. section 3.2). In § 24 this representation of the comparison particle by a scale is developed further, ignoring the existence of the comparison particle altogether. The use of a perfect object system, and the use of a scale are really the same thing from slightly different standpoints. What Eddington appears to have been unaware of is the fact that the construction of a perfect object-system leads very directly to the algebraic structures employed in the second half of **FT**. We propose to explain how this is so at some length, because it provides an important bridge between the two parts of **FT** which Eddington was himself unable to provide. The following argument is due mainly to Bastin (**24**).

6.2. We first define an 'observable' to be the result of a physical observation. Further we make, in accordance with the ideas of Chapter 1,

Axiom 1. An observable can be represented by an ordered set of n real numbers, or 'vector', in the sense of modern algebra, over the field, $\mathscr{R}$, of real numbers.

If observables exist which do not satisfy Axiom 1, our theory does not cover them. The number n is the order of the observable. The 'value' of an observable, ϕ, is the set of values of the n ϕ_α $(\alpha = 1,\ldots,n)$. An observable is thus completely specified when one is given:

(i) the experimental method of determination,
(ii) the method of calculation from experimental results,
(iii) the value.

We may note that, in order that the object system may be perfect, the ϕ_α must include the scale, describing the comparison particle. This point is further discussed in section 6.3.

The set of all observables of order n will be written O_n.

We now consider the formulation of the mathematical scheme corresponding to comparison of observables. From such statements as '*AB* is twice *BC*', or 'the momentum of P is at right angles to that of Q', we are led to define a comparison as follows:

If ψ, χ are two n-vectors and $\psi_\alpha = \sum_\beta P_{\alpha\beta}\chi_\beta$, and if ψ is in O_n if and only if χ is in O_n, P is a comparison of ψ and χ.

It follows that P is an *observable* of order n^2. O_n is clearly a subspace of V_n, the vector space of n dimensions (where the term subspace is not meant to exclude identity) and the comparisons of order n^2 are the linear mappings of O_n on to itself.

Thus from a set of observables of order n (> 1), we can form in turn comparisons of order n^2, n^4,..., n^k $(k = 2m$, m integral).

Now comparisons of order n^2 belong to the total matric algebra of order n^2 over $\mathscr{R}$. Further, the matric algebra of order n^4 is the direct product of two matric subalgebras, of orders n^2, so that the matric algebra of order n^4 suffices to represent both comparisons of order n^2 and comparisons between them.

We may regard the matrix product of two matrices, which represent comparisons, as representing the two comparisons in succession. Hence transformations of the algebra of comparisons which do *not* leave unchanged equations of the form

$$PQ = R,$$

where P, Q, R are comparisons, will be regarded as changes in the physical system examined. Accordingly, we define 'allowable transformations' of comparisons to be linear transformations leaving invariant equations of this form. It follows that transformations which are only changes in the reference system, or say, 'rotations', and not changes in the physical objects, will be allowable transformations. It does not follow that all allowable transformations are merely rotations.

6.3. The allowable transformations are, by definition, automorphisms of the algebra. Since a total matric algebra is normal simple, its automorphisms are inner automorphisms

$$P \to P' = qPq^{-1},$$

where q is a non-singular $n \times n$ matrix. It is convenient, here and later, to use the same symbol for an observable and its matrix representation. We shall also have to consider, however, subalgebras of observables, which may not be normal simple. In this case, the transformation will still have the same form provided that we allow q to belong to an appropriate algebra, which will have the P-algebra as a subalgebra. The measurement conditions for the simple case of a total matric algebra will be called the 'simple measurement conditions'. In this case, such transformations form an (n^2-1)-parameter group, $\mathscr{G}_N$ say, where

$$N = n^2-1.$$

This restriction of allowable transformations to automorphisms forms the first measurement condition.

Now we return to the need to include the scale, which we put in the form of the assumption that to every vector there corresponds a certain real number, derived linearly from the vector, which provides the magnitude of the scale. Further, this magnitude, by definition, will remain unchanged under those automorphisms which are only changes in the reference system. To change the scale is to change the method of measurement, and hence, from our point of view, the system measured. Algebraically, these considerations lead to the following axiom, which is really a restriction on the group of permitted automorphisms:

Axiom 2. There exists a linear mapping, σ, of the algebra of observables, $\mathscr{C}$, on to the field $\mathscr{R}$, which is invariant under those automorphisms which are only changes in the reference system. This mapping satisfies two conditions to be specified shortly.

Thus we have, for all P in $\mathscr{C}$,

$$\sigma(P) = p, \text{ say} \quad (p \text{ in } \mathscr{R}),$$

where $$\sigma(P+Q) = \sigma(P)+\sigma(Q)$$

and $$\sigma(\lambda P) = \lambda\sigma(P) \quad (\lambda \text{ in } \mathscr{R}).$$

$\sigma(P)$ is the scale of P.

The first condition arises in the following manner: if $P \rightarrow P^k$ is any automorphism of $\mathscr{C}$, then we say that σ is invariant under k if

$$\sigma(P^k) = \sigma(P) \quad (\text{all } P \text{ in } \mathscr{C}).$$

Thus σ defines a subgroup $\mathscr{G}^{(\sigma)}$ of $\mathscr{G}_N$ under which it is invariant. Axiom 2 asserts that the members of $\mathscr{G}^{(\sigma)}$ are the transformations which are only changes in the reference system. Hence we require that $\mathscr{G}^{(\sigma)}$ should not be a group with no members. Further, since we postulated Axiom 2 as a restriction on $\mathscr{G}_N$, we require that $\mathscr{G}^{(\sigma)}$ should not be the whole group. A theory in which all automorphisms are scale-preserving is of an algebraically simple kind, not corresponding to a physical theory since the system has then lost all contact with the uranoid (the scale is merely formal). Hence

First scale condition. A scale σ determines a scale-group $\mathscr{G}^{(\sigma)}$, which is a proper, non-zero subgroup of $\mathscr{G}_N$.

The second scale condition arises in the following manner: there will exist a set of linear mappings α_i ($i = 1,..., \rho$ say) of $\mathscr{C}$ on to $\mathscr{R}$, which are invariant under the whole group $\mathscr{G}_N$. For a total matric algebra there is essentially only one such, the trace. However, to cover the case of subalgebras, as above, we include the whole set of α_i. Now if σ is any mapping satisfying Axiom 2 and the first scale condition, so also is

$$\sigma' = \lambda_0\sigma + \sum_{i=1}^{\rho} \lambda_i \alpha_i.$$

This is an equivalence relation between σ and σ', and we refer to σ, σ' as equivalent scales, and write $\sigma \equiv \sigma'$. Clearly $\mathscr{G}^{(\sigma)} = \mathscr{G}^{(\sigma')}$; we cannot thus expect $\mathscr{G}^{(\sigma)}$ to determine the scale of which it is the scale-group. But we can postulate

Second scale condition. A scale-group $\mathscr{G}^{(\sigma)}$ determines the scale of which it is a group, up to equivalence.

Thus sets of equivalent scales, and scale-groups, determine each other. A theory in which two non-equivalent linear mappings σ, σ' are invariant under a scale-group $\mathscr{G}^{(\sigma)}$ is logically possible, but corresponds to a needless complication of the algebra. The object of a discussion like the present is to present the simplest theory which expresses the physical operations of measurement.

6.4. Consider now the case when $\mathscr{C}$ is a total matric algebra. Clearly there exists a fixed matrix B in $\mathscr{C}$ such that

$$\sigma(P) = \operatorname{tr}(BP),$$

and we speak of the scale-operator B. Even if $\mathscr{C}$ is not a total matric algebra, it is a subalgebra of a total matric algebra. In this case the equation is still true, provided that we take B in the total algebra. For a rotation we have, for all P,

$$\operatorname{tr}(BqPq^{-1}) = \operatorname{tr}(BP),$$

whence

$$Bq = qB.$$

Hence the group $\mathscr{G}(B)$ of 'rotations', or scale-preserving automorphisms, corresponds to the commutator algebra of the scale-operator B. Now if q commutes with B, it commutes also with B^2. Thus B^2 is another possible scale-operator, so that, by the scale conditions,

$$B^2 = \lambda B + \mu.$$

If $4\mu+\lambda^2 \neq 0$, and

$$B' = \frac{(B-\frac{1}{2}\lambda)}{(\pm[\mu+\frac{1}{4}\lambda^2])^{\frac{1}{2}}}$$

the sign being chosen for reality, then

$$B'^2 = \pm 1.$$

If $4\mu+\lambda^2 = 0$, then

$$B^2 = 0.$$

Hence it is sufficiently general to assume the scale-operator to be nilpotent, or to have unit square (apart from sign).

Returning to the commutation equation, we note that if

$$B^2 = \pm 1,$$

we can write any matrix q in the form

$$q = q_1 + q_2,$$

where

$$q_1 B = Bq_1, \qquad q_2 B = -Bq_2.$$

For we have

$$2q_1 = q \pm BqB,$$

$$2q_2 = q \mp BqB.$$

If

$$B^2 = 0$$

this decomposition is no longer true. Now the scale-operator was originally arbitrary; Axiom 2 restricted us to a choice between $B^2 = \pm 1$ or 0. The most convenient choice is clearly $B^2 = \pm 1$. A theory with nilpotent scales is a logical possibility, but we shall confine ourselves here to the non-singular case. Accordingly:

Axiom 3. $B^2 = \pm 1$.

We can now determine the order of $\mathscr{G}(B)$ in terms of $\operatorname{tr} B = s$, say. (We use the term order applied to a finite continuous group to denote the number of essential parameters.) If $B^2 = 1$ then $\frac{1}{2}(1-B)$ is idempotent, and hence, under automorphisms,

$$\tfrac{1}{2}(1-B) \simeq \begin{bmatrix} I_r & \cdot \\ \cdot & \cdot \end{bmatrix}$$

(**25**, p. 88). Here $\quad s = n-2r, \qquad r = \frac{1}{2}(n-s).$

But if q commutes with B, it commutes with $\frac{1}{2}(1-B)$, so that

$$\begin{bmatrix} I_r & \cdot \\ \cdot & \cdot \end{bmatrix} q = q \begin{bmatrix} I_r & \cdot \\ \cdot & \cdot \end{bmatrix}$$

which at once shows the order of the commutator algebra to be

$$r^2+(n-r)^2.$$

Taking account of the unit element, which belongs to the commutator algebra, $\mathscr{G}(B)$ is seen to be a κ-parameter group, where

$$\begin{aligned} \kappa &= r^2+(n-r)^2-1 \\ &= \tfrac{1}{2}(n^2+s^2)-1. \end{aligned}$$

Similarly, if $B^2 = -1$,
we have $$\kappa = \tfrac{1}{2}(n^2-s^2)-1.$$

The group $\mathscr{G}(B)$ has a one-parameter subgroup $\mathscr{G}_1$, given by, say,

$$q = \cos\tfrac{1}{2}\theta + B\sin\tfrac{1}{2}\theta \qquad (B^2 = -1),$$

or by $$q = \cosh\tfrac{1}{2}u + B\sinh\tfrac{1}{2}u \quad (B^2 = 1).$$

Leaving this aside, a group $\mathscr{G}_{\kappa-1}$ remains. The second measurement condition restricts us to this subgroup of the group of automorphisms.

6.5. Since the scale is to define the magnitudes of the elements of a measurable vector, there must exist some algebraic relations between the elements, to provide a means of comparison. Such a relation is provided by the minimum equation of the matric algebra

$$g(P) \equiv \sum_{i=0}^{r} a_i P^i = 0.$$

Since P can satisfy no equation of lower degree, any other such relation contains $g(P)$ as a factor. Hence it is sufficiently general to adopt this equation as scale-measuring relation. The mathematical formulation thus satisfies the third measurement condition automatically.

6.6. We have now specified the simple measurement conditions. They have led to the postulation of a scale-operator B, with square ± 1. We can define a linear set $\mathscr{L}$, whose elements are $\sum \lambda_i q_i B q_i^{-1}$. This linear set is contained in $\mathscr{C}$ and is therefore finite. It generates an algebra $\mathscr{S}$ say, which we call the scale-algebra of B. Clearly $\mathscr{S} \subseteq \mathscr{C}$. In the particular case when $\mathscr{S} = \mathscr{C}$, we call B an enveloping scale-operator.

In such a case $\mathscr{C}$ has a basis $e_0, e_1,..., e_N$, where e_0 is the unit element, generated by all finite products of q elements $e_1, e_2,..., e_q$ such that

(i) e_i^2 is scalar ($i = 1,..., q$),
(ii) e_i, e_j are equivalent under automorphisms, for any $i, j = 1,..., q$.

Thus any of the elements e_i ($i = 1,..., q$) could be taken as scale-operator. Suppose, in particular, that we can choose $e_1,..., e_q$ in such a way that there is a complete symmetry in the basis, i.e. so that any of the elements $e_1, e_2,..., e_N$ could be taken as a scale-operator. We cannot impose conditions (i) and (ii) as they stand, for if $e_m = e_i e_j$ is an element of the basis, and the square of every element is $x = \pm 1$, we have

$$e_m^2 = e_i e_j e_i e_j = x,$$

so that

$$x e_j e_i = e_i e_j,$$

and every pair of elements either commute or anticommute. The second case restricts us to quaternions alone, and in the first case the algebra is abelian, which is impossible for a total matric algebra. Instead, then, we define two quantities e_i, e_j to be pseudo-equivalent under automorphisms if either

$$e_i = \pm q e_j q^{-1},$$

or

$$e_i = \pm i q e_j q^{-1}.$$

We then define a scale-basis as a basis $e_1,..., e_N$ generated by all finite products of a set $e_1,..., e_q$ such that

(i) $e_i^2 = \pm 1$ ($i = 1,..., N$),
(ii) any two e_i, e_j are pseudo-equivalent under automorphisms,
(iii) the set $e_1, e_2,..., e_N$ is linearly independent.

The last restriction is equivalent to the requirement that there is a minimal set of $e_1, e_2,..., e_q$ which just generates the algebra. Any two elements of a scale-basis will then commute or anticommute. Further, since at least one element of the basis is the product of two anticommuting elements, e_i, e_j say, and since

$$\mathrm{tr}(e_i e_j) = \mathrm{tr}(e_j e_i) = -\mathrm{tr}(e_i e_j),$$

this element, and hence by (ii), all elements, will have zero trace. Thus their commutator algebras are of order $\frac{1}{2}n^2$.

A necessary and sufficient condition for a total matric algebra to admit a scale-basis is that it should be an even Clifford algebra, $\mathscr{C}_{2r}$ say, that is, the algebra generated by $2r$ anticommuting elements (**26**). For the number of products of the set $e_1, e_2, \ldots, e_q$ is

$$1+q+\frac{q(q-1)}{2!}+\ldots = 2^q.$$

But since it is a total matric algebra, 2^q is square, so that q is even, $= 2r$ say. We have thus the 2^r-rowed matric algebra, which is known to be the even Clifford algebra, $\mathscr{C}_{2r}$.

It is easy to verify that for $\mathscr{C}_{2r}$ the basis elements do indeed define scales satisfying Axiom 2 and the scale conditions. If we take the real basis, Newman (**27**, **28**) shows the set $e_1, \ldots, e_{2r+1}$, where $e_{2r+1} = e_1 e_2 \ldots e_{2r}$, to have P, M elements with positive, negative squares, where

$$P-M \equiv 1 \pmod{8}.$$

But
$$P+M = 2r+1,$$

so that
$$P \equiv r+1, \qquad M \equiv r \pmod{4}.$$

We may write
$$P = 4x+r+1, \qquad M = -4x+r,$$

where x is an integer satisfying

$$-\tfrac{1}{4}(r+1) \leqslant x \leqslant \tfrac{1}{4}r.$$

We may thus expect the even Clifford algebras to be of fundamental importance in this theory. Since the E-number algebra is $\mathscr{C}_4$, and the EF-algebra is $\mathscr{C}_8$, the concept of scale invariance provides the explanation of the use of these algebras in fundamental theory.

6.7. Returning to Eddington's own use of his ideas, the importance of perfecting the object system is now connected with the fluctuation of the scale. When the comparison particle is outside the object it is a mean particle, and so has a very small scale fluctuation based on the distribution of all the particles of the uranoid, but when the comparison particle is introduced into the object system it becomes an individual particle and so has a scale fluctuation $\sqrt{N}$ times as great. This is a very large scale uncertainty, but it is held by Eddington not to affect external particles whose masses are found indirectly. It does affect internal particles, and it is stated that the analysis usually employed gets over this difficulty by replacing this large uncertainty by a distribution over discrete eigen-scales; no reason is given for this belief.

Eddington concludes § 23 with a rather brief discussion (helpfully amplified in the 1943 draft, S, p. 135) of the peculiar 'energy tensor'

$Ap_\mu p_\nu$ which he employs. We have already noticed his inconsistency in mixing scale-free and scale-fixed characteristics, and his remarks do nothing to rectify this. But he does show that he realizes the strangeness of a resolution into densities in the ratio m^2/m'^2 of particles whose masses are m, m'. He concludes that he is considering a special distribution in which there is not a one:one correspondence between object-particles and comparison particles per unit volume. The occupation factor is proportional to the mass of the particles; no physical reason is given why this should be so. This seems to show that even if our special interpretation (section 4.7) is a useful one, Eddington himself was unaware of this way of looking at the relation.

6.8. As we have remarked § 23 of **FT** begins the third Chapter of an earlier draft; in **FT**, however, § 24 begins Chapter 3. The reason for this is clear; the perfect object-system elaborated in § 23 is only to be considered by using the extra variable ('scale') to describe the environment, but in the next few sections this is going to be done in the more complex abstract structure appropriate to quantum mechanics, in which the scale, amongst other variables, has a conjugate variable. Eddington first remarks that wave mechanics usually postulates flat space. The reference to the diagram of Chapter 1 will explain his contention that it would be confusing to try to extend wave mechanics to a curved space. The situation is that general relativity and quantum mechanics are in different sequences.

Eddington now proposes to take account of the phenomena described by curvature by introducing an explicit scale uncertainty. This is in keeping with the fundamental relation which he derived in his first Chapter. The assumption is, in detail, that a system requires an additional variable, a scale, with its probability distribution, and in keeping with quantum mechanics there will be a conjugate variable which he calls phase. He finds it most convenient to take the phase, as in our case, to be a coordinate since the scale may reduce to an eigenvalue. We then have a geometrical representation in five dimensions. The way in which the scale actually measures some characteristic (the 'scale indicator') is that we have a constant quantity whose measure exhibits the fluctuations of the standard. (In the case of ordinary quantities these fluctuations are concealed by the fluctuations of the quantity being measured.) The scale momentum then has a probability distribution with standard deviation σ_ϵ. We must be clear about the kind of scale considered here. In an earlier draft (**S**, p. 13) Eddington discusses the 'molar scale' which makes the spherical curvature of a section of space-time the standard of length, and so reduces the law of gravitation to an identity. This molar scale is the result of averaging the scale now being considered over a large number of particles. If we are considering a very small number of particles the fluctuations in this scale may be important.

6.9. The alternative formulation uses an angular momentum instead of a momentum as the scale, and this has the advantage that the conjugate phase has a distribution between 0 and 2π. This consideration of angular variables which can be restricted to their proper values is held by Eddington to be the basis of quantization (see **S**, p. 84). This fact is glanced at in **FT** where it takes the form of a phase which has a linear coordinate which is limited to a range of length $2\pi l$. A Fourier expansion then provides scale momenta of magnitude $n\hbar/l$. § 24 concludes with a curious argument about comparing spherical space and flat space. The basis of this argument is merely good fortune as is noted by Eddington on the top of the next page, since it depends on identifying two different radii.

6.10. The next paragraph makes a fresh beginning, and points out that observable coordinates involve, in virtue of the principle of relativity, two physical entities. With the other two entities which furnish the standard a complete measurement has a quadruple probability distribution. Eddington then assumes that the particular simplicity involved in a scientific investigation is that defined by measurements in which this distribution degenerates into a double or a single distribution. This appears to us to be a very small assumption which may be correct, but which requires justification.

Eddington's next step, however, is based on one of the most difficult ideas in **FT**, not least because it bears considerable resemblance to an orthodox idea, that of exchange energy—while it is essentially different. The difference does not seem to have been fully realized by Eddington, though it will be quite clear to anyone reading his work. Taking the four entities which constitute a measurable, he remarks that frequently one uses a mean scale or a physical origin, or both, to reduce the probability distribution to a double one; but, in general, one has a quadruple probability distribution. He considers then a proton, an electron, and their two comparison particles, and discusses the transformation in which the two comparison particles are interchanged. There is no distinction between comparison particles, so observable results must be invariant for this interchange. In this respect, interchange is completely different from exchange phenomena in quantum mechanics, which depend explicitly on the similarity of the particles which are exchanged. Of course, there is some connexion between the two. If the particles *are* similar and we identify them mathematically by chalking numbers on them, the observable results will be invariant for interchange of these numbers, and we can make this interchange into the sort of interchange which we have discussed already. But even for protons and electrons, interchange in Eddington's sense is still possible. If we do put suffixes on the particles, we can envisage a perfect object particle as having its scale and phase and its suffix carried by the comparison particle.

6.11. Eddington then considers the analysis of a system between proton and one electron in terms of the two-particle transformation. The analysis is actually carried out in the next paragraph, § 26. He quotes the usual classical formulae

$$M = m+m', \qquad \mu = \frac{mm'}{m+m'},$$

and remarks that we cannot take this into quantum mechanics or relativity theory without another investigation. As far as relativity theory is concerned, it is very hard to see how we can have any two-particle transformation at all. The difficulty arises from the different proper times to be attached to the two particles, as is well known. However, Eddington observes that in quantum mechanics a formal relativistic two-particle transformation can be set up and this is a remarkable result. If

$$X^\alpha = \frac{mx^\alpha + m'x^{\alpha'}}{m+m'}, \qquad \xi^\alpha = x^\alpha - x^{\alpha'},$$

we have easily

$$\frac{1}{m}\frac{\partial^2}{\partial x^\alpha \partial x^\beta} + \frac{1}{m'}\frac{\partial^2}{\partial x^{\alpha'}\partial x^{\beta'}} = \frac{1}{M}\frac{\partial^2}{\partial X^\alpha \partial X^\beta} + \frac{1}{\mu}\frac{\partial^2}{\partial \xi^\alpha \partial \xi^\beta},$$

which we may write symbolically as

$$\frac{p^\alpha p^\beta}{m} + \frac{p^{\alpha'} p^{\beta'}}{m'} = \frac{P^\alpha P^\beta}{M} + \frac{\varpi^\alpha \varpi^\beta}{\mu}.$$

However, these results are only true with the symbolic interpretation of the momenta, so that we have here left the domain of ordinary relativity theory. This fact is recognized again by Eddington at the end of the paragraph when he wishes to use numerical values for the momenta and employs the non-relativistic formulae for the energies. He says, what is of course well known for the classical two-particle transformation, that the energy of the internal and external particles together form the total energy. In the earlier draft of 1943, he discusses at some length the difference between the approximate value of the energy and the accurate one, but both there and later he considers that each of these values is the correct one for some part of the theory and that it is not the case that one is an approximation to the other. He also discusses why it should be the case that one form of the energy is additive while the other is not, but his arguments are not easy to follow.

6.12. Now, when the proton and electron are discussed in this way, Eddington claims that one comparison particle is eliminated. But this comparison particle, as well as carrying scale, carries the suffix, and so we must take account of the possible exchange of suffixes explicitly if we drop the comparison particle. This he proposes to do by introducing a permutation variable, and he further decides to use for this purpose a continuous variable θ such that interchange corresponds to the addition of π to θ. Then the addition of 2π to θ restores the situation, so that the permutation coordinate is like a phase. The coordinate is unobservable, but the constant angular momentum conjugate to it gives a term in the energy which is the interchange energy. This is the energy which Eddington claims to be the Coulomb energy. The idea of a continuous permutation coordinate has been formed by Eddington from that of a discrete interchange by analogy with the way in which wave mechanics makes continuous what had earlier been described as quantum jumps. The importance of such an idea is to be seen, for instance, in the fact that for an ordinary plane of rotation one has symmetrical degeneracy. This state of affairs will also arise with the interchange rotation.

6.13. The application of these ideas to the Coulomb energy occurs in § 33. This paragraph is an exceedingly difficult one, a fact which was noted by E. A. Milne (**29**). Eddington begins by observing that an interchange of the particles reverses the sign of ξ^{α}. He concludes that the Coulomb energy will appear in the equations of the intracule, but not of the extracule. He concedes that an adequate treatment of the Coulomb energy needs the algebraic theory, but he attempts to explain in general terms the way it arises. By an explanation in general terms he means one which shows how some such term might arise without providing an adequate theory of the factor $1/r$. In his wave-mechanical theory this factor arises in a detailed argument which involves the Jacobian of certain transformations. Here there is only an approximate argument, the $1/r$ being a factor which converts angular into linear momentum.

The first part of the paragraph is a rather lengthy piece of special pleading for the introduction of an angular momentum of magnitude of $\hbar$ instead of $\frac{1}{2}\hbar$ as one would expect. This factor 2 gave Eddington some trouble, as can be seen by the totally different explanations of it in the earlier drafts. In any case, it seems hardly worth spending much effort on at the moment when the rest of the Coulomb expression is so difficult to derive.

The next part of his argument involves the multiplicity of 137, and this is based on § 19, with which, as we have said, we cannot agree at all. Thus, most of the argument for the magnitude for the coupling between the electron and the electromagnetic field falls to the ground; but there is still the possibility that the factor $1/r$ might be derivable in our theory. As we have said, this factor arises when one converts angular into linear momentum, but the argument for this given by Eddington is so puzzling that we are forced to leave the matter open here.

In conclusion, then, we are, as regards the Coulomb energy, able to substantiate or agree with most of Eddington's argument up to the last part, but are completely puzzled over the final steps. This is particularly unfortunate as Eddington regarded the identity of the Coulomb energy with the interchange energy as the distinctive feature of his theory, at least in an early manuscript. He says there: 'It seemed clear at the time that those who chose to disregard it were following the wrong track in their attempts to relate quantum to relativity theory; so that from that date the author's researches have been uninfluenced by what is often called the "accepted" relativistic quantum theory—accepted in the sense that it is considered important to discover why it breaks down.'

7

PROPER MASS

7.1. Eddington begins his Chapter 4 with three sections not closely connected with the succeeding ones (see Appendix I). Then in § 37 he begins a development which might well have been placed in his § 1. This development involves considerable difficulty and repays an investigation of its earlier history. At some time before July 1942 (S, C, 33, i, §§ 4, 5) he gave the following argument:

The wave functions ψ, χ for the geometrical coordinates of the physical origin and the object have Fourier integral forms, which produce for the product function

$$\psi(x_0)\chi(x) = \frac{1}{2\pi\hbar}\iint e^{(i/\hbar)(p_0x_0+px)}\Psi(p_0)\mathrm{X}(p)\,dp_0\,dp,$$

to describe the combined system. We can choose new coordinates

$$\xi = x-x_0, \qquad \eta = \tfrac{1}{2}(x+x_0)$$

(or, equivalently, $\eta = \dfrac{m_0x_0+mx}{m+m_0}$ since this gives identical results) and define $p_0x_0+px = p_1\xi+p_2\eta$. Then the wave function for the combined system is

$$h(\xi,\eta) \equiv \psi(x_0)\chi(x) = \frac{1}{2\pi\hbar}\iint e^{(i/\hbar)(p_1\xi+p_2\eta)}\Psi(\tfrac{1}{2}p_2-p_1)\mathrm{X}(\tfrac{1}{2}p_2+p_1)\,dp_1\,dp_2.$$

Eddington now assumes that for the ξ-distribution alone (i.e. the relative coordinate) we have a wave function

$$h(\xi) = \int h(\xi,\eta)\,d\eta = \frac{1}{\sqrt{(2\pi\hbar)}}\int e^{ip_1\xi/\hbar}H(p_1)\,dp_1 \quad \text{(say)}.$$

This gives, by using the formula

$$\int e^{(i/\hbar)\alpha x}\,d\alpha = 2\pi\hbar\,\delta(x) = 2\pi\,\delta\!\left(\frac{x}{\hbar}\right),$$

$$H(p_1) = \sqrt{(2\pi\hbar)}\Psi(-p_1)\mathrm{X}(p_1),$$

so that $$|H(p_1)|^2\,dp = |\mathrm{X}(p)|^2w(p)\,dp,$$

where $w(p) = 2\pi\hbar|\Psi(-p)|^2$. That is, he concludes that the 'geometrical momentum distribution' is converted to a 'physical momentum distribution' by a weight function $w(p)$. Since, moreover, the frequency function of the origin has the form (by section 2.1) $(2\pi\sigma^2)^{-\frac{1}{2}}\, e^{-\frac{1}{2}x_0^2/\sigma^2}\, dx_0$, he assumes a wave function for the origin of the form

$$\psi(x_0) = (2\pi\sigma^2)^{-\frac{1}{4}}\, e^{-\frac{1}{4}x_0^2/\sigma^2},$$

so that $w(p)$ has also a Gaussian form with a standard deviation $\varpi = \frac{1}{2}\hbar/\sigma$. This choice of wave function for the origin is a very special one, since it neglects the phase; but if we choose an arbitrary phase the result will be very different. However, we can probably concede to Eddington that, at least for an order-of-magnitude argument, the phase will not make much difference. What then appears from the argument is the convergence factor to cut off large momenta, which is known in field theory.

Unfortunately the crucial assumption, that

$$h(\xi) = \int h(\xi, \eta)\, d\eta$$

is quite clearly false; for since $h(\xi, \eta)$ is a wave function this formula permits the interference of waves differing suitably in phase. But this is not the way in which we can ignore one coordinate. (Our attention was drawn to this fact by Dr. P. W. Higgs.) All that we can say is that

$$|h(\xi)|^2 = \int |h(\xi, \eta)|^2\, d\eta.$$

The fact that interference occurred was already known to Eddington in the early draft, but he seems to have convinced himself at that time that it was correct. By August 1943, however (**S**, **H**, 8, iv, §§ 31, 32), he was aware of the phase difficulty; but he explained incoherent phases as caused by the inclusion of several independent particles, so that for some origin the phase of $\Psi(p)$ is constant for all p when we have a 'symmetrical wave' for one 'fully observed particle'.

Finally in **FT** Eddington devotes § 37 to the corresponding theory when ψ, χ are replaced by frequency functions f, g, and defers the consideration of wave functions till § 38. The working is, of course, formally correct for frequency functions but its interpretation is not at all clear since these functions are not used in the orthodox theory. We can try to carry out the working suggested by Eddington in a correct fashion; we then have

$$\begin{aligned} |h(\xi)|^2 &= \frac{1}{(2\pi\hbar)^2} \int\!\!\int\!\!\int\!\!\int\!\!\int e^{(i/\hbar)[(p_1-\bar{p}_1)\xi+(p_2-\bar{p}_2)\eta]}\Psi(\tfrac{1}{2}p_2-p_1)X(\tfrac{1}{2}p_2+p_1)\times \\ &\qquad \times\Psi^*(\tfrac{1}{2}\bar{p}_2-\bar{p}_1)X^*(\tfrac{1}{2}\bar{p}_2+\bar{p}_1)\, dp_1\, dp_2\, d\bar{p}_1\, d\bar{p}_2\, d\eta \\ &= \frac{1}{2\pi\hbar} \int\!\!\int\!\!\int e^{(i/\hbar)(p_1-\bar{p}_1)\xi}\Psi(\tfrac{1}{2}p_2-p_1)X(\tfrac{1}{2}p_2+p_1)\times \\ &\qquad \times\Psi^*(\tfrac{1}{2}p_2-\bar{p}_1)X^*(\tfrac{1}{2}p_2+\bar{p}_1)\, dp_1\, dp_2\, d\bar{p}_1. \end{aligned}$$

This differs from the previous answer in having the variable p_2 present.

7.2. Accordingly in **FT**, § 38, Eddington goes on to consider wave functions, and imposes the convention that $|h(\xi)|^2$ has the form which gives the weight function. This means that if

$$K(p_1, \bar{p}_1, p_2) = \Psi(\tfrac{1}{2}p_2 - p_1)\Psi^*(\tfrac{1}{2}p_2 - \bar{p}_1)\mathrm{X}(\tfrac{1}{2}p_2 + p_1)\mathrm{X}^*(\tfrac{1}{2}p_2 + \bar{p}_1),$$

then $$\int K(p_2)\,dp_2 = K(0).$$

Since Ψ describes the object, which is arbitrary, Eddington decides to put the blame on to X, and require that all states which make an important contribution have $p_2 = 0$. (He expresses this in terms of a δ-function of p_2, having used a two-particle transformation.) The integration over p_2 then disappears and the result is that

$$\begin{aligned}|h(\xi)|^2 &= \frac{1}{2\pi\hbar}\int\!\!\int e^{(i/\hbar)(p_1-\bar{p}_1)\xi}\Psi(-p_1)\Psi^*(-\bar{p}_1)\mathrm{X}(p_1)\mathrm{X}^*(\bar{p}_1)\,dp_1\,d\bar{p}_1\\ &= \left|\frac{1}{\sqrt{(2\pi\hbar)}}\int e^{(i/\hbar)p\xi}\Psi(-p)\mathrm{X}(p)\,dp\right|^2.\end{aligned}$$

Even now we have not proved the earlier formula because of further phase difficulties; but at least the previous result is consistent with this one.

For this reason Eddington claims that $p_2 = 0$ for all important states is 'an implicit convention' of wave mechanics. To substantiate this he argues that the convention arises when the auxiliary apparatus of wave functions is used to analyse data consisting wholly of observables; 'possibly by a confusion of ideas' p_2 has been left out of the equations as physically meaningless. This is harmless if in fact $p_2 = 0$. Eddington interprets the convention physically by observing that then $p_0 + p = 0$, i.e. that every momentum p communicated to an object produces a recoil momentum $-p$.

It is instructive to see how the difficulties of these two sections are again those of a jump between abstract structures. The result which Eddington seeks to prove is provable without additional assumptions in the very simple structure in which the uncertainties of observation can be represented completely by frequency functions (i.e. classical physics). The 'convention' about p_2 is a device to simplify the more complex abstract structure, which allows of wave mechanics, so as to have the same result.

7.3. The analysis of the previous sections is now put on one side by Eddington, though the result about weight functions is quoted. It will be useful to introduce the discussion of **FT**, § 39, by considering the beginning of **FT**, § 27. The first paragraph of § 27 deals with mass constants: the remainder is outside our scope as it is simply an investigation of β-factors. He considers a particle energy ('hamiltonian') of the form

$$E = \mu_1 + \frac{p^2}{2\mu_2}$$

calling μ_1 the rest-mass and μ_2 the mass constant. These are clearly equal for a free particle. In a suitable field, however, they can be different. For example, as in Chapter 5, for the type II Lorentz invariant gravitational field we have

$$E = m_0(1-\psi) + \frac{p^2}{2m_0},$$

or

$$E = m_0(1-\psi) + \frac{p^2}{2m_0(1-\psi)^2},$$

according as p is defined as $m_0 v$ or mv. From Eddington's nomenclature it seems as if the first possibility is the one he had in mind, since μ_2 is then indeed constant. In each case $|\mu_1-\mu_2| = m_0\psi$ approximately. Similar considerations apply to the other type II field, which is interpreted as nuclear. Eddington asserts in **FT** that the same result is true for an electromagnetic field, but this is a mistake on his part. For in the electromagnetic field the rest-mass is not affected by the potential (because the force is automatically orthogonal to the four-velocity). This affects Eddington's definitions of *free* and *bound* particles by the equations $\mu_1 = \mu_2$ (so that $\psi = 0$), $\mu_1 = 0$ (so that $\psi = +\infty$ in the gravitational case, since $1-\psi$ is only an approximation for $e^{-\psi}$, but $\psi = 1$ in the nuclear case). These definitions do not consider the effect of the electromagnetic field. § 39 now deals with a very interesting and curious speculation about the rest-mass. We shall show that this speculation is also bedevilled by the difficulty of moving between abstract structures. The contention of the section is that in a uranoid a lowering of temperature produces a 'concealed' energy in the form of proper mass. Here again a historical approach will be helpful. In an earlier draft (S, C, 33, i, § 8) Eddington gives a rough calculation involving 'Newtonian' theory. He first says that the gravitational potential ϕ produces a rest-mass or rest-energy of a particle of 'mass-constant' m of amount

$$m' = m(1-\phi)$$

(in suitable units). The equality of rest-mass and rest-energy and the emphasis on rest implies that this Newtonian theory must really be in special relativity. We have just seen that two simple Lorentz invariant theories gave a change of rest-mass with potential energy, either of the form $m = m_0 e^{-\psi}$, or $m = m_0 - e\psi$. The function ψ is not of the same kind in the two cases; the first one is Eddington's ϕ, and the second is $m_0 \phi/e$. In each case, then, for weak fields we have an approximation

$$m = m_0(1-\phi).$$

In each of these theories we have field equations of the form

$$F_\mu = \partial_\mu \psi, \qquad \partial_\mu F^\mu = 4\pi\rho,$$

with a static solution
$$\psi = \int \frac{\rho}{r}\, d\tau.$$

If, then, lowering the temperature is to increase the object-mass m, and so decrease ψ, it is clear that this lowering must decrease the background density ρ. This is plausible so long as ρ denotes the 'rest energy and kinetic energy' density. Now, from the equations, ρ is an invariant, i.e. it is the rest-energy density *alone*; so that these Lorentz invariant theories cannot be the ones intended. On the other hand, if ρ *is* taken as the kinetic energy density, so that

$$\rho\, d\tau = \beta(\rho_0\, d\tau_0) \quad \text{and} \quad d\tau = \frac{d\tau_0}{\beta},$$

we have
$$\rho = \beta^2\rho_0 \doteqdot \rho_0 + \rho_0 v^2 = \rho_0 + 3P,$$

by the usual kinetic theory, where P is the pressure, and so

$$\rho = \rho_0(1+AT),$$

where A is a constant. Thus ρ increases with temperature and in fact

$$\frac{3P}{\rho_0} = \frac{\rho-\rho_0}{\rho_0} = AT.$$

Although this argument is superseded in **FT** it is very instructive because it is, on the one hand, a difficulty of the sort we have been describing involving a jump between abstract structures; but, on the other hand, it is a situation which is familiar in (e.g.) Einstein's early investigations in general relativity (see **30**) when a tentative argument mixing Lorentz-invariant and inconsistent Newtonian results gives a result which is then found to be consistently incorporated in a larger theory (for Einstein's arguments, in general relativity).

7.4. The actual argument of **FT**, § 39, is based on the quantity

$$\Theta = \frac{\rho-\rho_0}{\rho_0} = \frac{3P}{\rho_0}$$

which our tentative investigation leads us to identify with absolute temperature. In the Einstein universe we have

$$8\pi\gamma P = \lambda - \frac{1}{R_0^2}, \qquad 8\pi\gamma\rho = \frac{3}{R_0^2} - \lambda$$

so that

$$\Theta = \frac{3}{2}\,\frac{\lambda - 1/R_0^2}{3/R_0^2 - 2\lambda}.$$

Eddington notices the interesting fact that as λ, the cosmical constant, varies, so also does Θ. In fact the usual choice of $\lambda = 1/R_0^2$, $P = 0$ is the *zero-temperature uranoid,* and as λ increases to $3/2R_0^2$, Θ increases indefinitely, until we get the *infinite temperature uranoid.*

That this is a reasonable definition now follows because it involves $\rho_0 = 0$. But the usual definition of an infinite temperature assembly involves a distribution of momenta with a constant frequency function, i.e. in one variable

$$dN = a\,dp$$

is the number of particles with momenta between p and $p+dp$, where a is constant. This is of the form

$$dN = \frac{am_0\,dv}{(1-v^2)^{\frac{3}{2}}}$$

and since $\int\limits_{v_0}^{1} \frac{dv}{(1-v^2)^{\frac{3}{2}}}$ diverges this means that almost all the particles must have unit velocity, and therefore zero rest-mass. The two uranoids have

	Θ	λ	ρ	ρ_0	P
Zero temperature	0	$1/R_0^2$	$1/R_0^2$	$1/R_0^2$	0
Infinite temperature	∞	$3/2R_0^2$	$3/4R_0^2$	0	$1/4R_0^2$

the last three columns being in units in which the factor $4\pi\gamma$ has been removed.

The momenta involved in the definition of infinite temperature must be geometrical momenta (since they lead to infinite integrals). The corresponding physical momenta, if we accept the weight function argument for a moment, will be distributed according to $w\,dp$. The weight function w has standard deviation $\varpi = \hbar/2\sigma$, i.e.

$$w(p) = \frac{1}{\sqrt{(2\pi\varpi^2)}}\,e^{-p^2/2\varpi^2}.$$

Hence $\overline{p^2} = \varpi^2$. Since the rest-masses are zero the energy E is given by $E^2 - p^2 = 0$. But $p^2 = p_1^2 + p_2^2 + p_3^2$, and so $\overline{E^2} = 3\varpi^2$.

7.5. The last part of **FT**, § 39, contains a derivation of some constants in which it is again necessary to use the formula

$$T_{\mu\nu} = Ap_\mu p_\nu$$

for the energy tensor. The remarks we have made in Chapter 4 all apply here. For the infinite temperature uranoid, if A is treated as a constant, we have, for mean values

$$\bar{T}_{11} = \bar{T}_{22} = \bar{T}_{33} = A\varpi^2, \qquad \bar{T}_{44} = 3A\varpi^2,$$

so that for N particles

$$P = NA\varpi^2, \qquad \rho = 3NA\varpi^2.$$

On the other hand, in the zero-temperature case $\rho = NAm^2$, where m is the rest-mass. Hence

$$\frac{3\varpi^2}{m^2} = \frac{3}{4},$$

i.e.

$$m = 2\varpi = \frac{\hbar}{\sigma}.$$

Moreover,

$$4\pi\gamma NA\frac{\hbar^2}{\sigma^2} = \frac{1}{\mathrm{R}^2},$$

so that

$$A = \left(\frac{\sigma^2}{\mathrm{R}^2}\right)\frac{1}{4\pi\gamma\hbar^2 N}.$$

The results of Chapter 2 give us some value of σ/R; e.g. for flat-space $\sigma = \mathrm{R}/\sqrt{(5N)}$, so that $A = 1/(20\pi\gamma\hbar^2N^2)$. If we try to salvage this argument of Eddington's by defining a conventional energy tensor, we find that in the zero-temperature case this is not difficult, since

$$T_{\mu\nu} = \rho v_\mu v_\nu = np_\mu p_\nu(p_\alpha p^\alpha)^{-\frac{1}{2}},$$

where n is a number-density. But this gives $\rho = Nm$, as one would expect, and not $\rho = ANm^2$. It is much harder to carry this amendment out for the infinite temperature case, since p_μ is nul, and so must be normalized with some arbitrary time-like vector.

It is clear, then, that if this working of Eddington's is to be preserved it requires some interpretation like the one we have given in § 4.7 in which the relation $T_{\mu\nu} = Ap_\mu p_\nu$ is a rule connecting scale-free and scale-fixed analyses. Eddington's argument then gives a relation between σ and the scale-fixed mass.

The next section (**FT**, § 40) continues the discussion of the value of m, refining it by considering β-factors.

7.6. It will probably be sufficient to consider the remaining sections of **FT** Chapter 4 more briefly. In § 41 the value of ϖ (or equivalently m)

found above is applied in a zero-temperature uranoid where an exclusion principle operates. In § 42 this argument is continued and compared with the Dirac hole theory. (It must be remembered that in 1945 and earlier the hole theory was much more a living idea than now.) Finally in § 43 it is claimed that by means of the exclusion principle the same standard mass m is calculated as in the gravitational theory. This forms the basis of Eddington's contention that 'exclusion is a wave-mechanical substitute for gravitation'. This contention is amplified in an earlier draft, as Slater points out (**S**, **H**, 8, iv, § 35) the gravitational field concerned is the one usually concerned in wave-mechanics, i.e. the inertial field of the uranoid which produces the rest-masses. Then in §§ 44, 45 the theory of super-dense hydrogen is deduced in much the same way as that of the uranoid in § 41. We do not feel it worth while to treat these sections at length because on the one hand they depend extensively on what has been done before, much of which is in doubt, and on the other they form rather a technical investigation in a special sphere. If the underlying theory could be clarified, as it may be in due course, these detailed investigations can then be carried out.

8

CONCLUSIONS

8.1. We have now completed our description of Eddington's statistical theory. In this chapter we shall begin by trying to summarize our description briefly and then try to assess the value of this theory and of others like it. There are two unifying themes which run through the whole of the rather disjointed cavalcade of results of **FT**. One of these, a very obvious one, is the insistence on the consideration of the environment of the physical system considered. By far the most important aspect of this is the study of the scale, a property of the system which it possesses only by virtue of its relation to its environment. Since quantum mechanics is concerned with very small systems it is clear that the environment—at least to the extent that it determines scale—will be particularly important there. The projected summary (**FT**, p. 264) of the concluding chapters of **FT** shows that Eddington was well aware of the importance of scale in his theory.

The second theme does not appear to have been consciously formulated by Eddington. It is provided by the existence of the abstract structures of Chapter 1. Such ideas had clearly been in Eddington's mind for a long time, certainly since 1924 (see, for instance, pp. 237–8 of **11**). By 1936 he was using arguments which could only be based on some such structures, and on their *independent existence* in the sense that the simpler ones do not rely on more complex ones to which they may be considered approximations (though these may be a help in discussing the simpler ones). For example, § 6.3 of (**8**) is a discussion of a formalism said to describe one particle, using as a guide 'actual' space-time involving a large number. We are naturally not concerned here with the correctness or otherwise of that theory; but we are interested in Eddington's outlook. He argues in the section cited that as there is only one particle, which is charged, we have *A universe containing . . . only one sign of charge, . . . lopsided compared with a universe containing equal numbers of positive and negative charges . . . the enormous negative-potential would so alter the world that we should require a different type of space-time to frame the phenomena. It is just such a world in miniature, that our theory of a single charged particle imitates.*

It is natural to ask whether this second theme is of great importance in Eddington's work if he himself only formulated it unconsciously. The answer is that its importance lies in its enabling the requirements of the first theme to be carried out. The idea of a system with and without its environment is only logically possible in terms of abstract structures of the sort described. Similarly it is only in such terms that we can understand how the environment can be described satisfactorily in terms of one number, say the scale.

8.2. With these two themes in mind it is possible to summarize what has gone before, and we shall do so in what appears to be a more logical order than Eddington's (which was mainly followed in the rest of the book). There must be many other ways of putting together these sections which are equally satisfactory; it will suffice to give one.

We begin with section 6.1 describing in the most direct, physical manner the need for considering scale, and then sections 6.2–6.6 which show how this consideration of scale leads to Eddington's algebraic theories. Having introduced scale in this way, we take a somewhat more general view; clearly scale was just one example of considering environment. Now, at least in relativity, there is an intimate connexion between environment and the existence of a coordinate-system (for example curvature prevents the assumption of a universal cartesian coordinate-system). The next step is therefore to see how to use the coordinate-system to describe the environment, and especially to relate this to the scale-variable description already employed. For this purpose we make the hypothesis of the physical origin (section 2.3), leading to a parameter σ which accordingly will serve as a scale constant. Then the argument of section 2.4 will serve to connect this small length σ with a large one R. It is safest to choose the case of a sphere of N similar particles of radius R, and so have

$$\sigma = \frac{\mathrm{R}}{\sqrt{(5N)}}. \tag{53}$$

To relate this more closely to measured quantities we then use the physical coordinate-system to discuss scattering of charged particles. Until now we have not specified the nature of the particles considered at all. We wish to formulate the simplest possible abstract structure capable of containing Eddington's ideas, and so we postulate that all the particles are of the same mass m. Then in the discussion of sections 5.1–5.8 we find the result

$$r_0 = 1{\cdot}70\sigma, \tag{54}$$

where r_0 is the scattering range constant. We must notice here that these particles are charged, but equation (53) applies to neutral particles, so we must postulate that pairs of particles of opposite charge can be bound, the resulting mass being, of course, $2m$. Thus the total mass will be

$$M_0 = 2mN. \tag{55}$$

8.3. The relation (53) specifies a particular value for $\mathrm{R}/(\sigma\sqrt{N})$. For most purposes, however, N is considered infinite, and then there is no definite ratio of R and σ. In these circumstances we can make the division into scale-free and scale-fixed of section 3.3, and follow this with something equivalent to section 3.4. It is not very satisfactory to use section 3.4 as it stands because it is written (following Eddington) in terms of wave functions. A wave function description is quite certainly of a higher complexity than anything which has occurred up to now. But a glance at section 3.4 will show that this is only a way of expressing things; this section (and correspondingly **FT**, § 10) could be rewritten without mentioning wave functions at all. From this modified section 3.4 we then have the result (in very much the same way as in **FT**, § 10) that scale-free matter should, just as usual, be described by an energy tensor, and that scale-fixed particles are to be described by a momentum vector.

The question arises, how to establish a relation between scale-fixed and scale-free descriptions of the same situation. If the scale is to describe the whole environment, that is, if one number is to be sufficient to specify the 'size' of the system, it is clear that some special system of units must have been chosen with two relations between the units of mass, length, and time. One obvious candidate for such a relation is $c = 1$, since this has long been used to emphasize the essential identity of mass and energy in general relativity. If we are to have a theory involving quantum mechanics it seems certain that we should choose another relation involving $\hbar$. An obvious choice would be $\hbar = 1$ which is often adopted in practice; and this would be very simple because length, mass, and time would then all have the same dimension. However, this choice would give an energy tensor, in cartesian-like coordinates, the dimension $M/L^3 \equiv L^{-2}$, and a momentum vector the dimension $M \equiv L$. The fact that these indices are of opposite sign indicates that any connexion between $T_{\mu\nu}$ and p_μ will involve dimensional constants. We are at this stage free to choose our special units, and formulate our theory accordingly. We may note (cf. section 3.2) that Eddington's contention that the choice is free because it makes no difference is probably wrong; but it is free because we are constructing an arbitrary theory. (Some choices will give better systems than others, in the sense of experimental agreement.) If we choose $M/L^3 \equiv M^2$, i.e. $ML^2 = 1$, a choice which can conveniently be specified by Eddington's equation

$$8\pi\gamma\hbar^2 = 1,$$

then $T_{\mu\nu}$ and $p_\mu p_\nu$ have the same dimensions. With this choice we can make Eddington's fundamental postulate

$$T_{\mu\nu} = Ap_\mu p_\nu$$

or more accurately

$$T_{\mu\nu} = A \int n(p_\mu)p_\mu p_\nu \, d^4p,$$

with A a dimensionless constant, for the connexion between scale-fixed and scale-free analyses, as in section 4.7. Then, as in section 4.7, we can prove that the particles involved in the scale-fixed analysis have masses inversely proportional to their degrees of freedom.

We may remark that the fundamental postulate here has more of the status of a definition, since we are not claiming any particular physical significance for our scale-fixed particles, but only for the abstract structure as a whole. It is not clear, however, to what extent the same justification can be made for the extended postulate for the mutual energy tensor:

$$M_{\mu\nu} = \tfrac{1}{2}C(p_\mu p'_\nu + p'_\mu p_\nu),$$

or

$$M_{\mu\nu} = \tfrac{1}{2}C \iint n(p_\mu, p'_\mu)(p_\mu p'_\nu + p'_\mu p_\nu) \, d^4p d^4p',$$

where $n(p_\mu, p'_\mu)\, d^4pd^4p'$ is the number of pairs of particles of momenta p_μ, p'_μ respectively. If this extended postulate is made, it is quite easy to investigate what masses a scale-fixed particle must have to allow it to be inserted in a given environment, described by a particle of given mass m_0 (as in section 4.6). We find that only two masses are permitted, roots of

$$Am^2 + A'm_0^2 = Cmm_0,$$

where A, A', C depend on the number of degrees of freedom of the particles. There seems little difficulty in Eddington's own argument that $A:A':C = 10:1:136$, giving two masses in the ratio 1848:1, m_0 being about 136 times the smaller root. We must notice, however, that we have established no connexion between m_0 and the m of equation (55). Without this connexion the quadratic equation is only useful in determining the ratio of its roots. We can, if we like, keep matters at a somewhat simpler level than Eddington's by avoiding his 'reality arguments' and inquiring merely for the masses of particles described by a whole algebra. If the particles have k degrees of freedom, but the environment particle is constrained so that it has only one, we have

$$A:A':C = k:1:k^2$$

and the ratio is $\frac{1}{2}k(k^2+\Delta)-1$,

where $\Delta^2 = k^4-4k$. As soon as k is not too small this is approximately k^3; for $k = 4$ it is about 62.

8.4. The scale-fixed masses introduced by momentum vectors are intended to be quantum mechanical particles, so that one would expect some connexion with Planck's constant. Even without detailed working it seems certain that the mass introduced must be something whose Compton wavelength $\hbar/m_0$ is of the order of the small scale-constant σ. The same argument applied in the scattering case to indicate a connexion between $\hbar/m$ and σ means that m and m_0 are of the same order. We now have to go to the more complex description in terms of wave functions, as in sections 7.1–7.5. Since we are here dealing with neutral particles the formula in terms of the mass m of this chapter will be

$$\sigma = \hbar/2m. \tag{56}$$

It is therefore tempting to take $m = m_0 = \hbar/2\sigma$, and we shall make this assumption. This completes our description of the theory; those sections of **FT**, and of the present book, not included here are difficult to fit into the extremely simplified scheme we have given. It is quite likely that other schemes, little more complex, would include them but not all of our present results. We are concerned merely to exhibit one such.

8.5. We have now to try to assess the value of the structure we have outlined in terms of observations. The four equations (53)–(56) involve eight quantities only two of which (r_0, $\hbar$) can possibly be considered to be directly measured, even if we assume that the structure can be made to fit in such a way that they are the measured range-constant and Planck's constant. We therefore need some means of estimating the other quantities. These estimates will come from cosmological considerations, and it cannot make very much difference to the results which of a number of estimates we choose. (Indeed it is probably irrelevant, as explained by McCrea (**31**).) We shall therefore adopt much the same view as Eddington, and make the following assumptions:

1. The appropriate value of R to choose for our model is that of an expanding Eddington–Lemaître model:
2. Mass is conserved, so that the appropriate choice of M_0 is the initial Einstein mass

$$M_0 = \frac{\pi R_0}{2\gamma}, \tag{57}$$

where R_0 is the initial radius. We put $R = \alpha R_0$, and suppose α to be in the range from 1 to 10.

We can now estimate R and R_0 from the measured Hubble constant V since

$$V = \frac{1}{\surd 3 R_0}\sqrt{\left\{\frac{(\alpha-1)^2(\alpha+2)}{\alpha^3}\right\}}.$$

Even if α is as small as 2, V will be $0{\cdot}71/\surd 3\,R_0$, i.e. 70 per cent of its limiting value. We may therefore take the measured V as limiting, and so have

$$V_0 = \frac{1}{\surd 3\,R_0}. \tag{58}$$

The equations (53)–(58) now involve nine quantities of which four are directly measured; the remaining five may be eliminated, leaving one equation which provides an immediate estimate of the value of the theory. (In this statement we have ignored the parameter α, since it is of the order of unity.) Performing the elimination, writing $V_0 = 1/T$ and going over to c.g.s. units, the result is

$$1 = \frac{\alpha^2(1{\cdot}70)^3 2\surd 3\,.\,\hbar\gamma T}{15\pi r_0^3 c^2}.$$

If we use a value of $T = 4{\cdot}1\times 10^{17}$ sec (corresponding to a Hubble constant of 75 km/sec/megaparsec), the right-hand side is $1{\cdot}24\alpha^2$. When we consider the very simple nature of our abstract structure, this result can only be interpreted as indicating a very close agreement. (It should be remembered that the result might well have been wrong by a factor of 10^{10}.)

It is therefore instructive to use the equations to calculate the other quantities in the structure. We have, choosing as before

$$\hbar = 1{\cdot}05\times 10^{-27},$$
$$r_0 = 2{\cdot}1\times 10^{-13},$$
$$c = 3\times 10^{10},$$
$$T = 4{\cdot}1\times 10^{17},$$
$$\gamma = 6{\cdot}67\times 10^{-8},$$

that, in turn,

$$R_0 = 7{\cdot}1\times 10^{27},$$
$$M_0 = 1{\cdot}50\times 10^{56},$$
$$\sigma = 1{\cdot}24\times 10^{-13},$$
$$N = 5{\cdot}19\times 10^{80},$$

$m = 1{\cdot}44\times 10^{-25}$ ($\doteqdot$ 158·[illegible] electron masses)

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The argument in section 8.3 now gives, as the scale-fixed masses which can be inserted in the environment, approximately 2,155 and 1·166 electron masses. The surprising nearness of these to the proton and electron masses provides a confirmation of the value of our abstract structure, and also an indication that the argument of section 8.3 is probably sound.

8.6. We consider it fair to claim that the arguments we have used to establish the results of section 8.5 have very much less difficulty and obscurity than Eddington's original version. And, indeed, we believe that a more detailed investigation would remove what difficulties remain. Let us suppose this to be the case, and discuss the nature of the theory with which we are left.

It may be as well to make quite clear at once that it is not an *a priori* theory of the sort that Eddington claimed to be constructing. It is not unique; there have been numerous arbitrary choices in its construction, all of which have been made to the best of our ability with one eye on the experimental results. Other theories are perfectly possible, in which different experiments are given importance. And certainly more complex theories are possible, which will deal with more experimental results, and to greater accuracy.

In what respects does our abstract structure differ from one which might have been constructed from an orthodox point of view, seeking 'a rough approximation to the actual universe'? In the present state of our knowledge of abstract structures the answer must be: not very much. Perhaps the chief contribution of the ideas of Chapter 1 to the working of Chapter 8 is a certain confidence in approximating, knowing that some sort of abstract structure will result, which can then be compared with conventional experiments. But without such ideas Eddington cannot be understood at all.

The argument of the present chapter has, in common with Eddington, the difficulty of taking place in two abstract structures of differing complexities. We have glossed over the difficulty of comparing results in the structure involving wave functions with those in the simpler structure by making the identification $m = m_0$. By keeping the structures as simple as possible we were able to reduce this difficulty to that of making only one *ad hoc* assumption. Naturally the difficulty will usually be greater. This focuses attention on what appears to us to be the one problem whose solution will really contribute to progress in this field, instead of re-discussion and improvement of theories already put forward by Eddington. This problem is to determine the exact relationship between two abstract structures, one of greater complexity than the other. Slightly less ambitiously we could ask at least for the exact relations between numerical quantities associated with the structures. For example, both general relativity and Newtonian gravitation employ the same γ not merely approximately, but exactly the same. But the number of degrees of freedom of (e.g.) the gravitational potentials in the two theories are 10 and 1 respectively. Thus some numbers are invariant, and others are not. An even less ambitious problem would be: Given two abstract structures, one more complex than the other, to determine the numerical invariants of the homomorphism which carries one into the other.

To answer these problems in any general way it seems necessary to start with the simplest abstract structures, rather than the very complex ones which are ordinary scientific theories. The only attempts which have been made on this problem are due to Bastin (**32**) in Cambridge. Bastin considers an analogue machine of an unusual kind (a self-organizing mechanism) as a means of simulating simple abstract structures of differing complexities. There are reasons for thinking that experiments with machines of this kind will provide the necessary step forward but the work is still in progress, and is in any case outside the scope of this book.

APPENDIX I

FT, § 6

THE first part of § 6 of **FT** is devoted to a rather naïve geometrical picture of the results of **FT**, § 3. Eddington projects the points of spherical space orthogonally on to the tangent flat space at the origin and finds that the standard deviation of a coordinate of a particle with uniform probability distribution over the hypersphere is $\frac{1}{2}\mathrm{R}_0$, so that the standard deviation of a coordinate of the centroid of N particles is $\mathrm{R}_0/2\sqrt{N}$. Eddington remarks that orthogonal projection is preferred only in the light of the results of § 3; central or stereographic projection would give different results for the standard deviation. In section 2.4 we obtained the value of σ as $\mathrm{R}_0/1{\cdot}761\sqrt{N}$, so Eddington's preference for orthogonal projection is unjustified.

Eddington then continues by warning us against confusing the $g_{\mu\nu}$ of general relativity with the $g_{\mu\nu}$ that arises through writing the wave equation in tensor form, since these are two entirely different quantities. The ideas expressed are similar to those of **FT**, § 13.

FT, § 17

Eddington introduces the *complete momentum vector* with 10 independent components and the *complete energy tensor* with 136 independent components. These are fully investigated later in **FT**, but here he uses these ideas in dealing with the multiplicity factors of certain particles.

He defines a particle as a 'conceptual carrier of a set of variates', and uses the term 'carrier' as an alternative to particle. This definition applies to composite as well as elementary particles, but the more complex carriers are generally referred to as systems. Eddington asserts that discovering a particle means observing certain effects which are accepted as proof of its existence, so that if we invent particles to carry a set of variates which arise in the analysis, then there are no difficulties as to whether or not the particle really exists. Experimental test is not concerned with the particle, but with the variates that it carries. Hence, when we consider a set of variates, we consider that they are carried by some particle.

The simplest kind of carrier in scale-free physics is a carrier of a complete energy tensor and nothing more. If the energy tensor is not subject to stabilizing conditions, the carrier is called a *standard carrier*. In this case the number of independent components of the energy tensor is 136 so that the standard carrier is a V_{136}.

An ordinary energy tensor may be put in the form

$$T^{\mu\nu} = \rho_0 v^{\mu} u^{\nu} + S^{\mu\nu},$$

where ρ_0 is the proper density, v^{μ} the velocity vector of the external motion, and $S^{\mu\nu}$ is the internal stress system. When $S^{\mu\nu} = 0$, $T^{\mu\nu}$ reduces to the outer square of a vector,

i.e. $$T^{\mu\nu} = v^{\mu}\sqrt{\rho_0}\,.\,v^{\nu}\sqrt{\rho_0}.$$

Eddington calls $v^{\mu}\sqrt{\rho_0}$ the *root vector*. This is, of course, the correct form of Eddington's equation $T^{\mu\nu} = p^{\mu}p^{\nu}$, since the quantities on both sides of the equation are scale-free. However, Eddington makes the very dubious assertion that when the transition is made from scale-free to quantum physics, the present scale-free particles are given a fixed scale, and the root vector will become the recognized momentum vector of the scale-fixed particle.

In order to keep with scale-free physics, we prefer to use the complete root vector with ten independent components. This may be considered to consist of a 4-vector representing linear velocity and a 6-vector representing angular velocity. Thus, in scale-free physics we have a choice between V_{136} particles, or standard carriers, characterized by an unrestricted energy tensor and V_{10} particles, or vector carriers, characterized by a complete root vector.

Vector carriers are obtained from standard carriers by imposing the stabilizing condition that the energy tensor is the outer square of a vector. Further stabilization leads to the V_3 and V_4 particles mentioned in **FT**, § 11.

FT, § 21

The rigid-field theory discussed in section 4.2 does not lead, as Eddington's does in **FT**, § 15, to a necessary change of sign between particle and total energy. This change of sign is taken up by Eddington in **FT**, § 21, and he relates it to the occurrence of the complex number field in quantum mechanics. This relation depends on the argument that an imaginary velocity iv is needed to produce negative kinetic energy; if mv is then redefined as momentum, a factor i will be needed in the formulae. The curiously intimate way in which the complex number field is involved in quantum mechanics certainly raises problems of interest. But, especially because of our amendments to the arguments of **FT**, it does not seem that Eddington's explanation goes very far towards a solution.

FT, § 34

This section begins with a discussion of the effective mass, in general relativity, of a system of particles in motion. The working appears, according to the Editorial note in **FT**, to be wrong, but as Slater remarks (**S**, p. 36) the result is not used, and all that is needed is the standard result from wave mechanics that a Gaussian wave packet with standard deviation σ has at time t a standard deviation $\sigma(t)$ given by

$$[\sigma(t)]^2 = \sigma^2+\frac{\hbar^2 t^2}{4m^2\sigma^2}.$$

FT, §§ 35, 36

These two sections deal with some very interesting ideas of Eddington, which we have brought to the Appendix only because they are not connected with the rest of our book. All information about a system must be of particular states of the system in interaction with its environment. By an 'undisturbed' state—Eddington would prefer the term *normal*—we must mean one in which the system has only undergone an agreed amount of disturbance by probing. Sometimes we may want to incorporate fresh information; to do this we must redefine the normal state to allow more probing.

To give a general theory, Eddington suggests that (analogously to the replacing of a complex environment by a simple one, scale) we should consider a simple specification of keenness of probing. First we list how many pairs of conjugate variables, (x_α, p_α), we seek to measure. One such pair is *fully observed* if the combined uncertainty is just the Heisenberg limit. Suppose that at time $t = 0$ we cease to probe; at time $t > 0$ the pair is less than fully observed (see note above on **FT**, § 34) and so t acts as a *coefficient of under-observation.*

Eddington argues that such a coefficient is an observable for the system, and so subject to uncertainty. And he contends that the symmetry between this coefficient and the space coordinates accounts for the success of 'relativistic wave mechanics'; because it makes it sensible to consider probability distributions over four variables. But this fourth variable is confused with the time, which increases with it, in orthodox quantum mechanics.

FT, §§ 46–48

In these three sections Eddington discusses the planoid and the special planoid. The planoid is simply the sphere of particles which we have employed, particularly in Chapter 8, for formulating an abstract structure. Eddington prefers to define the planoid to have the same uncertainty as his uranoid so that he has the equation

$$\frac{\mathrm{R}_1^2}{N_1} = \frac{5}{4}\frac{\mathrm{R}_0^2}{N},$$

which with Slater's correction would read

$$\frac{\mathrm{R}_1^2}{N_1} = \frac{5}{9}\frac{\mathrm{R}_0^2}{N}.$$

He introduces, however, the special planoid with the same R_0 and N as the uranoid; this is the one we have employed. He is able to say fairly comprehensively how various numerical quantities transform between the uranoid and the two planoids, but this is because of the formulae relating these quantities to each other and to R_0 and N, that he has already found. It would be very difficult to do this otherwise; indeed it is the same problem of differing abstract structures which we have met in many other places.

APPENDIX II

THE DENSITY–PARTICLE CORRELATION

THE difficulties surrounding Eddington's equation

$$T^{\mu\nu} = p^{\mu}p^{\nu}$$

have been discussed. However, this equation has been used as the basis of an idea known as the density–particle correlation. This has been fully described by the authors in two papers (**33**, **34**); we will here outline the idea and its results.

The equation can be rewritten in the form

$$T^{\mu\nu} = C\frac{c^4}{8\pi\gamma\hbar^2}p^{\mu}p^{\nu}, \tag{59}$$

where C is a dimensionless constant. However, it suffers from the disadvantage that p^{μ} is defined on the world-line of a particle, and $T^{\mu\nu}$ is defined everywhere. To overcome this the following definition is made.

Consider an assembly of equal particles in a Riemannian space-time $\bar{S}$; we may approximately consider the particles to define a momentum-vector p^{μ} at every point of $\bar{S}$. Then $T^{\mu\nu}$ defined by (59) has the dimensions of an energy tensor. Subject to certain conditions (which are investigated), $T^{\mu\nu}$ will be the energy-tensor of a distribution of continuous matter in another space-time S (say). This correlation between S, $\bar{S}$ is called the *density–particle correlation.*

Certain restrictions are placed on the correlation and this leads to the spaces S, $\bar{S}$ being conformal. S is taken to be an expanding universe of general relativity, but the application of an overall consistency condition prohibits the expansion and leads to S, $\bar{S}$ being identical Einstein universes, which is exactly the case considered by Eddington.

Eddington intended the momenta in his equation to be defined quantum mechanically; i.e. $p_{\mu} = -i\hbar\Pi_{\mu} = -i\hbar\frac{\partial}{\partial x^{\mu}}$. However, to preserve the covariant nature of the theory, we define

$$-(i\hbar)^{-1}p^{\mu}\psi = \Pi^{\mu}\psi = g^{\mu\nu}\psi_{;\nu}.$$

Thus if we put

$$-\frac{8\pi\gamma}{c^4}T^{\mu\nu} = G^{\mu\nu},$$

where $G^{\mu\nu}$ is the Einstein tensor of the space S, and introduce an operand ψ, equation (59) becomes

$$G^{\mu\nu}\psi = C\Pi^{\mu}\Pi^{\nu}\psi.$$

It is found that ψ must be a scalar. The covariant differentiation on the right-hand side is carried out in $\bar{S}$.

This equation can be written in the form

$$G_{\mu\nu}\psi = C\psi_{;\mu\nu}. \tag{60}$$

When S and $\bar{S}$ are both Einstein universes, the solution for ψ is found to be time-dependent only.

In an attempt to generalize this solution, the equation (60) was adopted as the basis of the density–particle correlation with no restrictions. This was considered in an expanding universe and led to several solutions for ψ, the most interesting being

$$\psi = H\frac{1-r^2/4}{1+r^2/4}\mathrm{R}_0\cosh\frac{ct}{\mathrm{R}_0}-M\sinh\frac{ct}{\mathrm{R}_0}$$

corresponding to a standard metric of the form

$$ds^2 = c^2\,dt^2-\frac{\mathrm{R}^2(t)}{(1+r^2/4)^2}(dx^2+dy^2+dz^2)$$

with $$\mathrm{R} = \mathrm{R}_0\cosh\frac{ct}{\mathrm{R}_0},$$

where R_0 is the radius of the Einstein universe and H and M are arbitrary constants. This corresponds to a model expanding from an initial Einstein state, and so may be considered the most immediate generalization of Eddington's model. It requires $C = -2$.

Suppose that this function is in some respects analogous to a ψ-function in the quantum-mechanical sense. Accordingly it is natural to consider a distribution of particles with density ψ^2 per unit volume and we use this as a means for defining the uncertainty of the origin.

It is found that the standard deviation of the distribution lies between $\mathrm{R}/2{\cdot}482$ and $\mathrm{R}/1{\cdot}197$, so that at all times and for all values of the arbitrary constants, Eddington's uncertainty constant lies between $\mathrm{R}/2{\cdot}482\sqrt{N}$ and $\mathrm{R}/1{\cdot}197\sqrt{N}$.

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Eddington's 'Fundamental Theory' has remained an obscure and difficult book and still defies efforts to understand it, even far enough to make refutation possible. It falls into two parts; in the first (*the statistical theory*) certain numerical properties of scientific methods of description are related to fundamental constants of nature, and in the second (*the algebraic theory*) the values of these numerical properties are calculated. The present book presents a new criterion against which Eddington's theory can be judged, by explaining how it is possible for a correct theory of Eddington-type to exist. In the light of this criterion the arguments of the statistical theory are examined section by section

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